8.4 Matrix Exponential

Objectives:

- What is exponential function e^{At} where A is a matrix of constants?
- To see the answer of a question whether we can find a matrix exponential function e^{At}, where A is a matrix of constants, so that e^{At} is a solution of the system $X' = AX$.

Matrix Exponential: For any $n \times n$ matrix A, the matrix exponential is defined as

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots + A^k\frac{t^k}{k!} + \cdots = \sum_{k=0}^{\infty} A^k\frac{t^k}{k!}$$

Derivative of e^{At}:

$$\frac{d}{dt}e^{At} = Ae^{At}$$

- If $X' = AX$ is a homogeneous system, then $X = e^{At}C$ is a solution of this system, where A is an $n \times n$ matrix of constants and C is an $n \times 1$ column matrix of arbitrary constants.
- e^{At} is a fundamental matrix of the system $X' = AX$.
- If $X' = AX + F(t)$ is a non-homogeneous system, where A is an $n \times n$ matrix of constants, then

$$X = X_c + X_p = e^{At}C + e^{At}\int_{t_0}^{t} e^{-As}F(s)ds$$

is a solution of the non-homogeneous system.

Question 7/363: Find the general solution of the system

$$X' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} X \ .$$

Laplace Transform: The Laplace transform of a function $f(t)$ is

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)dt = \lim_{b\to\infty}\int_0^{b} e^{-st} f(t)dt,$$

provided that the integral converges.

In general, the Laplace transform of a function is a function of s, that is, $L[f(t)] = F(s)$.

Inverse Transform: If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $L[f(t)] = F(s)$, we say then $f(t)$ is the inverse Laplace transform of $F(s)$ and we write $f(t) = L^{-1}[F(s)]$.

Some Laplace and Inverse Transforms:

Laplace Transform	*Inverse Transform*
(a) $L[1] = \dfrac{1}{s}$	$1 = L^{-1}\left[\dfrac{1}{s}\right] =$
(b) $L[t^n] = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$	$t^n = L^{-1}\left[\dfrac{n!}{s^{n+1}}\right], \quad n = 1, 2, 3, \ldots$
(c) $L[e^{at}] = \dfrac{1}{s-a}$	$e^{at} = L^{-1}\left[\dfrac{1}{s-a}\right]$
(d) $L[\sin kt] = \dfrac{k}{s^2+k^2}$	$\sin kt = L^{-1}\left[\dfrac{k}{s^2+k^2}\right]$
(e) $L[\cos kt] = \dfrac{s}{s^2+k^2}$	$\cos kt = L^{-1}\left[\dfrac{s}{s^2+k^2}\right]$
(f) $L[\sinh kt] = \dfrac{k}{s^2-k^2}$	$\sinh kt = L^{-1}\left[\dfrac{k}{s^2-k^2}\right]$
(g) $L[\cosh kt] = \dfrac{k}{s^2-k^2}$	$\cosh kt = L^{-1}\left[\dfrac{k}{s^2-k^2}\right]$

- With the help of inverse Laplace transform, we find compute e^{At} as follows

$$e^{At} = L^{-1}\left[(sI - A)^{-1}\right],$$

where I is the identity matrix.