8.3.2 Non-Homogeneous Linear Systems (Variation of Parameters)

Objectives:

- To study the solutions of non-homogeneous linear system $X' = AX + F(t)$
- In particular, to find the particular solution of non-homogeneous linear system $X' = AX + F(t)$.

If $X_1, X_2, \ldots, X_n$ is a fundamental set of solutions of the homogeneous system $X' = AX$, then we know that its general solution is

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n \text{ or}$$

$$X = c_1 \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix} + c_1 \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix} + \cdots + c_1 \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}}_{\Phi(t)} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{C}$$

$\Phi(t)$ is called ***fundamental matrix***

- Consider the non-homogeneous system

$$X' = AX + F(t) \qquad (*)$$

- The particular solution of (*) is

$$X_p = \Phi(t)\int \Phi^{-1}(t)F(t)dt$$

Integrate term by term

- The general solution of (*) is

$X = X_c + X_p$ (where X_c is the solution of corresponding homogeneous system $X' = AX$) or

$$X = \Phi(t)C + \Phi(t)\int \Phi^{-1}(t)F(t)dt$$

Initial Value Problem: The solution of the IVP

$$X' = AX + F(t) \quad \text{subject to} \quad X(t_0) = X_0$$

is

$$X = \Phi(t)\Phi^{-1}(t_0)X_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)F(s)ds$$

Question 14/359: Use variation of parameters to solve the system

$$X' = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} X + \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix} e^{2t}.$$

Question 29/359: Use variation of parameters to solve the system

$$X' = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} X + \begin{bmatrix} e^{t} \\ e^{2t} \\ te^{3t} \end{bmatrix}.$$

Question 31/359: Solve the IVP

$$X' = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} X + \begin{bmatrix} 4e^{2t} \\ 4e^{4t} \end{bmatrix} \quad \text{subject to} \quad X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$