

8.2 Homogeneous Linear Systems

Objectives:

- To study the relation between eigenvalues (& eigenvectors) of the matrix A with solutions of homogeneous linear system $X' = AX$
- To solve homogeneous systems $X' = AX$ if
 - A has distinct real eigenvalues
 - A has distinct complex eigenvalues
 - A has both distinct real and complex eigenvalues

Observation: Relation of eigenvalues (& eigenvectors) with solutions of homogeneous systems

- Consider a homogeneous linear system

$$X' = AX \quad (*)$$

- Question: When a vector of the form $X = \mathbf{v}e^{\lambda t}$ will be a solution of (*)?

- Answer: $X = \mathbf{v}e^{\lambda t} \Rightarrow X' = \mathbf{v}\lambda e^{\lambda t}$

Using X, X' in (*) we see that

$$\mathbf{v}\lambda e^{\lambda t} = A\mathbf{v}e^{\lambda t} \Leftrightarrow \mathbf{v}\lambda = A\mathbf{v} \text{ or}$$

$$A\mathbf{v} = \lambda\mathbf{v}$$

$X = \mathbf{v}e^{\lambda t}$ is a solution of (*)



$$A\mathbf{v} = \lambda\mathbf{v} \Leftrightarrow (A - \lambda I)\mathbf{v} = 0$$

(i.e. λ is an eigenvalue of A and \mathbf{v} is an eigenvector of A)

By knowing eigenvalues & eigenvectors we can find solutions of (*)

We will study solutions of homogeneous systems as

- **distinct real** eigenvalues of A { **this section** }
- **distinct complex** eigenvalues of A { **this section** }
- **Repeated** eigenvalues of A { **next section** }

Solving homogeneous linear systems

Case I: Distinct real eigenvalues

- Given $X' = AX$ (*) ($A: n \times n$)

and A has n real distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

- We need n independent solutions
- Each eigenvalue will give one.

- ❖ For each eigenvalue λ_i , find an associated eigenvector \mathbf{v}_i

➤ The solution vector associated to this eigenvalue is $X_i = \mathbf{v}_i e^{\lambda_i t}$

- ❖ Find all solution vectors and write general solution as

$$X = c_1 X_1 + c_2 X_2 + \dots + c_n X_n$$

$$\text{or } X = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \dots + c_n \mathbf{v}_n e^{\lambda_n t}$$

Question 8/351: Find general solution of the system

$$\frac{dx}{dt} = 2x - y$$

$$\frac{dy}{dt} = 5x + 10y + 4z$$

$$\frac{dz}{dt} = 5y + 2z$$

Question 14/351: Solve the IVP

$$X' = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Solving homogeneous linear systems

Case II: Repeated eigenvalues

- Given $X' = AX$ (*)
and A has repeated eigenvalues.
- In such cases it may not always be possible to find linearly independent solutions of the homogeneous systems.
- If m is a positive integer and $(\lambda - \lambda_1)^m$ is a factor of the characteristic equation while $(\lambda - \lambda_1)^{m+1}$ is not a factor, then λ_1 is said to be an *eigenvalue of multiplicity* of m .

- In general, for some $n \times n$ matrices A it may be possible to find m linearly independent eigenvectors $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_m$ corresponding to an eigenvalue λ_1 of multiplicity $m \leq n$. In this case the general solution of the system contains the linear combination

$$c_1 \mathbf{V}_1 e^{\lambda_1 t} + c_2 \mathbf{V}_2 e^{\lambda_1 t} + \dots + c_m \mathbf{V}_m e^{\lambda_1 t}$$

- If there is only one eigenvector corresponding to the eigenvalue λ_1 of multiplicity m , then m linearly independent solutions of the form

$$X_1 = \mathbf{v}_{11} e^{\lambda_1 t}$$

$$X_2 = \mathbf{v}_{21} t e^{\lambda_1 t} + \mathbf{v}_{22} e^{\lambda_1 t}$$

⋮

$$X = \mathbf{v}_{m1} \frac{t^{m-1}}{(m-1)!} e^{\lambda_1 t} + \mathbf{v}_{m2} \frac{t^{m-2}}{(m-2)!} e^{\lambda_1 t} + \dots + \mathbf{v}_{mm} e^{\lambda_1 t}$$

Where \mathbf{V}_{ij} are column vectors, can always be found.

Procedure for Finding Linearly Independent Vectors when multiplicity of Eigenvalues is > 1

We consider a 3×3 matrix.

Step 1: Assume that matrix has 3 eigenvalues $\lambda_1, \lambda_1, \lambda_2$.

Step 2: Eigenvalues λ_1 has multiplicity “2” while λ_2 is unique.

Step 3: Two cases for λ_1

λ_1 has zero defect

(that is, for one eigenvalue we get two linearly independent eigenvectors.

Thus defect is: $2-2=0$

In this case the matrix admits two linearly independent vectors \mathbf{v}_1 and \mathbf{v}_2
Solution of the system for this eigenvalue is

$$X_1 = \mathbf{v}_1 e^{\lambda_1 t} \text{ and } X_2 = \mathbf{v}_2 e^{\lambda_1 t}$$

λ_1 has defect

(that is, for one eigenvalue we get one linearly independent eigenvector.

Thus defect is: $2-1=1$

In this case the matrix admits two linearly independent vectors \mathbf{v}_1 and \mathbf{v}_2
Solution of the system for this eigenvalue is

$$X_1 = \mathbf{v}_1 e^{\lambda_1 t} \text{ and } X_2 = (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda_1 t}$$

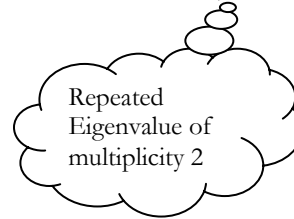
Example with Defect 1

Solve the system

$$x' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} x$$

Step1 Eigenvalues

$$\begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 6\lambda + 9 = 0 \Rightarrow \lambda = -3, -3$$



Step2 Eigenvectors with respect above Eigenvalue

- Use $\lambda_1 = -3$ in the matrix $(A - \lambda_1 I)$ and reduce it to Echelon form:

$$\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \xrightarrow{R_1/6} \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

- The Eigenvector is: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3r \\ r \end{pmatrix} = 3r \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

- One Solution is: $\vec{x} = v_1 e^{\lambda_1 t} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}$

No Second Eigenvector Exists.
Thus there is a defect: $2-1=1$

Step2 Procedure for **finding Second** linearly independent **Eigenvectors**

- Require that

- $(A - \lambda I)v_2 = v_1$

$$\Rightarrow (A - \lambda I)^2 v_2 = (A - \lambda I)v_1 \quad (1)$$

Use $(A - \lambda I)v_1 = 0$ above to get

$$(A - \lambda I)^2 v_2 = 0$$

- $(A - \lambda I)v_1 = 0 \quad (2)$

- Solve $(A - \lambda I)v_2 = v_1$ by reducing it to Echelon form

Summary
One Defect Case

- $(A - \lambda I)v_2 = v_1$
- $(A - \lambda I)v_1 = 0$

- Let $v_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. Then solving above equation is equivalent to reducing

- $\begin{pmatrix} 6 & -18 & | & 3 \\ 2 & -6 & | & 1 \end{pmatrix}$ to Echelon form.

$$\begin{pmatrix} 6 & -18 & | & 3 \\ 2 & -6 & | & 1 \end{pmatrix} \xrightarrow{R_1/6} \begin{pmatrix} 1 & -3 & | & 1/2 \\ 2 & -6 & | & 1 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & -3 & | & 1/2 \\ 0 & 0 & | & 0 \end{pmatrix}$$

- The Corresponding Eigenvector is:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3r \\ r \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = r \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

Set $r = 0$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

Step3 Second Solution is:

$$x_2 = (v_1 t + v_2) e^{\lambda t} = \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right] e^{-3t}$$

Step4 General Solution is:

$$\vec{x} = \left\{ c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right] \right\} e^{-3t}$$

Example with Defect zero

Solve the system $x' = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} x$

Step1: Eigenvalues

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0 \Rightarrow -3 \begin{vmatrix} 2 & 1-\lambda \\ -1 & -2 \end{vmatrix} + 6 \begin{vmatrix} -2-\lambda & 2 \\ -1 & -2 \end{vmatrix} - \lambda \begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$-3[-4+1-\lambda] + 6[4+2\lambda+2] - \lambda[\lambda^2 + \lambda - 2 - 4] = 0$$

$$3\lambda + 9 + 12\lambda + 36 - \lambda^3 - \lambda^2 + 6\lambda = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\lambda_{1,2} = -3, -3, \lambda_3 = 5$$

Step2: Eigenvectors

- Use $\lambda_1 = -3$ in $(A - \lambda_1 I)$ and reduce it to Echelon form:

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} \xrightarrow[\substack{R_2 - 2R_1 \\ R_3 + R_1}]{} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2r + 3s \\ r \\ s \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -r \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

Corresponding to Eigen value of multiplicity 2, we get two linearly independent vectors. Hence **defect is zero**

- Use $\lambda_1 = 5$ in $(A - \lambda_1 I)$ and reduce it to Echelon form:

$$\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{pmatrix} \xrightarrow{-1 \times R_1} \begin{pmatrix} 1 & 2 & 5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{pmatrix}$$

$$\xrightarrow[\substack{R_2 - 2R_1 \\ R_3 + 7R_1}]{R_2 / -8} \begin{pmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{pmatrix} \xrightarrow{R_3 / 16} \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[\substack{R_1 - 2R_2 \\ R_3 - R_2}]{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -r \\ -2r \\ r \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -r \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Step3 Solutions

Substitute the three vectors in $x_i = v_i e^{\lambda_i t}$ to get three solutions:

- $x_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} e^{-3t}$

- $x_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} e^{-3t}$

- $x_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} e^{5t}$

Step4 General Solution

$$x = c_1 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} e^{-3t} + c_3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} e^{5t}$$

Procedure for finding two missing Eigenvector when Eigenvalue has multiplicity 3
Two Defect Case

• **Finding Eigenvectors given v_1**

- $(A - \lambda I)v_3 = v_2$
- $(A - \lambda I)v_2 = v_1$
- $(A - \lambda I)v_1 = 0$

• **Three Corresponding Solutions**

- $x_1 = v_1 e^{\lambda t}$
- $x_2 = (v_1 t + v_2) e^{\lambda t}$
- $x_3 = \left(\frac{v_1 t^2}{2} + v_2 t + v_3 \right) e^{\lambda t}$

Example: Solve the system $x' = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} x$

Step1: Eigenvalues

$$\begin{vmatrix} 2 - \lambda & 1 & 6 \\ 0 & 2 - \lambda & 5 \\ 0 & 0 & 2 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda_{1,2,3} = 2$$

Step2: Eigenvectors (Reduce $(A - \lambda_1 I)$ and reduce it to Echelon form)

$$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2/5} \begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - 6R_2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We need three Eigen vectors but get only 1.
 The defect is $3-1=2$. To find two vectors we use
 $(A - \lambda I)v_3 = v_2$ and $(A - \lambda I)v_2 = v_1$ such

that $(A - \lambda I)v_1 = 0$

• **Finding v_2**

Reduce to Echelon form the matrix $((A - \lambda I)v_2 = v_1$:

$$\begin{pmatrix} 0 & 1 & 6 & | & 1 \\ 0 & 0 & 5 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{R_2/5} \begin{pmatrix} 0 & 1 & 6 & | & 1 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{R_1-6R_2} \begin{pmatrix} 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \\ 1 \\ 0 \end{pmatrix} = r \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ setting } r = 0$$

• **Finding v_3** Reduce to Echelon form the matrix $(A - \lambda I)v_3 = v_2$:

$$\begin{pmatrix} 0 & 1 & 6 & | & 0 \\ 0 & 0 & 5 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{R_2/5} \begin{pmatrix} 0 & 1 & 6 & | & 0 \\ 0 & 0 & 1 & | & 1/5 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{R_1-6R_2} \begin{pmatrix} 0 & 1 & 0 & | & -6/5 \\ 0 & 0 & 1 & | & 1/5 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \\ -6/5 \\ 1/5 \end{pmatrix} = r \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix} \text{ setting } r = 0$$

Step3: Solutions

• $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$

• $x_2 = (v_1 t + v_2) e^{2t} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] e^{2t}$

• $x_3 = \left(\frac{v_1 t^2}{2} + v_2 t + v_3 \right) e^{2t} = \left[\frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix} \right] e^{2t}$

Step4: General Solution

It is given by

$$x = c_1 x_1 + c_2 x_2 + c_3 x_3$$

Exercise 8.2.2

Q19 Find general solution of $x' = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} x$

Step1: Eigenvalues

$$\begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = 0 \Rightarrow \lambda = 0, 0$$

Step2: Eigenvectors corresponding to Eigenvalue "0"

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \xrightarrow{R_1/3} \begin{pmatrix} 1 & -1/3 \\ 9 & -3 \end{pmatrix} \xrightarrow{R_2-9R_1} \begin{pmatrix} 1 & -1/3 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r/3 \\ r \end{pmatrix} = r/3 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Deficit = 2-1=1. Hence use
 $(A - \lambda I)v_2 = v_1$

• **Second Eigenvectors using $(A - \lambda I)v_2 = v_1$**

Reduce $\begin{pmatrix} 3 & -1 & | & 1 \\ 9 & -3 & | & 3 \end{pmatrix}$ to Echelon form:

$$\begin{pmatrix} 3 & -1 & | & 1 \\ 9 & -3 & | & 3 \end{pmatrix} \xrightarrow{R_1/3} \begin{pmatrix} 1 & -1/3 & | & 1/3 \\ 9 & -3 & | & 3 \end{pmatrix} \xrightarrow{R_2-9R_1} \begin{pmatrix} 1 & -1/3 & | & 1/3 \\ 0 & 0 & | & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r/3 \\ r \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$$

Step3: Solutions

$$x_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$x_2 = (v_1 t + v_2) e^{\lambda t} = \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} \right]$$

Step4: General Solution

$$x = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} \right]$$

Q25 Find general solution of $x' = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix} x$

Step1: Eigenvalues

$$\begin{vmatrix} 5-\lambda & -4 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 5-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 5, 5, 0$$

Step2: Eigenvector when $\lambda = 5$

$$\begin{pmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -5 & 2 \\ 0 & -4 & 0 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{\substack{R_2 / -4 \\ R_3 / 2}} \begin{pmatrix} 1 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\substack{R_1 + 5R_2 \\ R_3 - R_2}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2r \\ 0 \\ r \end{pmatrix} = -r \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

We **need** two Eigen vectors but get only 1. This implies that the defect is $2-1=1$. To find second vector we use and $(A - \lambda I)v_2 = v_1$ such that $(A - \lambda I)v_1 = 0$

• **Finding Second Eigenvector when $\lambda = 5$ (use $(A - \lambda I)v_2 = v_1$)**

$$\begin{pmatrix} 0 & -4 & 0 & | & 2 \\ 1 & -5 & 2 & | & 0 \\ 0 & 2 & 0 & | & -1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -5 & 2 & | & 0 \\ 0 & -4 & 0 & | & 2 \\ 0 & 2 & 0 & | & -1 \end{pmatrix} \xrightarrow{\substack{R_2 / -4 \\ R_3 / 2}} \begin{pmatrix} 1 & -5 & 2 & | & 0 \\ 0 & 1 & 0 & | & -1/2 \\ 0 & 1 & 0 & | & -1/2 \end{pmatrix}$$

$$\xrightarrow{\substack{R_1 + 5R_2 \\ R_3 - R_2}} \begin{pmatrix} 1 & 0 & 2 & | & -5/2 \\ 0 & 1 & 0 & | & -1/2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2r - 5/2 \\ -1/2 \\ r \end{pmatrix} = -r \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} -5/2 \\ -1/2 \\ 0 \end{pmatrix} \Rightarrow$$

$$v_2 = \begin{pmatrix} -5/2 \\ -1/2 \\ 0 \end{pmatrix}$$

• Finding Second Eigenvector when $\lambda = 0$

$$\begin{aligned} & \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 2 \\ 5 & -4 & 0 \\ 0 & 2 & 5 \end{pmatrix} \xrightarrow{R_2 - 5R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -4 & -10 \\ 0 & 2 & 5 \end{pmatrix} \\ & \xrightarrow{R_2 \times -1/4} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5/2 \\ 0 & 2 & 5 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2r \\ -5/2r \\ r \end{pmatrix} \\ & \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -r/2 \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix} \end{aligned}$$

Step3: Solutions

$$x_1 = v_1 e^{5t} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} e^{5t}$$

$$x_2 = [v_1 t + v_2] e^{5t} = \left[\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} t + \begin{pmatrix} -5/2 \\ -1/2 \\ 0 \end{pmatrix} \right] e^{5t}$$

$$x_3 = v_3 e^{0t} = \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix}$$

Step4: General Solution

$$x_1 = \left\{ c_1 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + c_2 \left[\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} t + \begin{pmatrix} -5/2 \\ -1/2 \\ 0 \end{pmatrix} \right] \right\} e^{5t} + c_3 \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix}$$

**A word on Check that your Vectors and solutions are correct
The following must be satisfied**

$$\text{(Matrix of Eigenvector)}^{-1} \text{(Given matrix)} \text{(Matrix of Eigenvectors)} = \begin{pmatrix} 5: & \dots 1: & 0: \\ 0 & \dots 5: & 0: \\ 0 & \dots 0: & 0: \end{pmatrix}$$

Q29 Solve the IVP $x' = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} x$ with $x(0) = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

Step1: Eigenvalues

$$\begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 8\lambda + 16 = 0 \Rightarrow \lambda = 4, 4$$

Step2: Eigenvectors

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow -R_2} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \xrightarrow{R_2 + 2R_1} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2r \\ r \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = r \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We need two Eigen vectors but get only 1. This implies that the defect is $2-1=1$. To find second vector we use and $(A - \lambda I)v_2 = v_1$ such that $(A - \lambda I)v_1 = 0$

• **Finding Second Eigenvector (solve $(A - \lambda I)v_2 = v_1$)**

$$\begin{pmatrix} -2 & 4 & | & 2 \\ -1 & 2 & | & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow -R_2} \begin{pmatrix} 1 & -2 & | & -1 \\ -2 & 4 & | & 2 \end{pmatrix} \xrightarrow{R_2 + 2R_1} \begin{pmatrix} 1 & -2 & | & -1 \\ 0 & 0 & | & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2r - 1 \\ r \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = r \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Step3: Solutions

$$x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t}$$

$$x_2 = (v_1 t + v_2) e^{4t} = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{4t}$$

Step4: General Solution

$$x = \left\{ c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \right\} e^{4t}$$

Step5: Initial Condition $x(0) = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

Apply above condition in general solution to get:

$$\begin{pmatrix} -1 \\ 6 \end{pmatrix} = \left\{ c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$$

To determine constants reduce to Echelon form the following matrix

$$\left(\begin{array}{cc|c} 2 & -1 & -1 \\ 1 & 0 & 6 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 0 & 6 \\ 2 & -1 & -1 \end{array} \right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & -1 & -13 \end{array} \right)$$

$$\xrightarrow{R_2 \times -1} \left(\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 13 \end{array} \right) \Rightarrow c_1 = 6, \quad c_2 = 13$$

The solution of the IVP is:

$$x = \left\{ 6 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 13 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \right\} e^{4t}$$

Check that our answer is correct

Inverse $\left[\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} // \text{MatrixForm}$

$$\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$$

Solving homogeneous linear systems

Case III: Distinct complex eigenvalues

- Given $X' = AX$ (*)
and A has distinct complex eigenvalues
- Recall: complex eigenvalues and eigenvectors occur in conjugate pairs.

- ❖ For each pair $p \pm iq$ of eigenvalues, find the associated pair of eigenvectors and split into real and imaginary parts as $\mathbf{a} \pm i\mathbf{b}$
 - The solution vectors associated to the pair of eigenvalues $p \pm iq$ are

$$\begin{aligned} X_1 &= e^{pt} (\mathbf{a} \cos qt - \mathbf{b} \sin qt) \\ X_2 &= e^{pt} (\mathbf{b} \cos qt + \mathbf{a} \sin qt) \end{aligned}$$

- ❖ Find all solution vectors and write general solution.

Question 40/352: Find the general solution of the system

$$\frac{dx}{dt} = 2x + y + 2z$$

$$\frac{dy}{dt} = 3x + 6z$$

$$\frac{dz}{dt} = -4x - 3z$$

Solving homogeneous linear systems

Mixture of Case I and Case III

- Given $X' = AX$ (*)

and A has distinct real and complex eigenvalues

- ❖ For each real eigenvalue λ_i , find eigenvector \mathbf{v}_i

➤ Then $X_i = \mathbf{v}_i e^{\lambda_i t}$ is corresponding term in solution

- ❖ For each complex pair $p \pm iq$ of eigenvalues, find the pair of eigenvectors

$$\mathbf{a} \pm i\mathbf{b}$$

➤ Then

$$\begin{aligned} e^{pt} (\mathbf{a} \cos qt - \mathbf{b} \sin qt) \\ e^{pt} (\mathbf{b} \cos qt + \mathbf{a} \sin qt) \end{aligned}$$

are corresponding terms in the general solution

Question 44/352: Find the general solution of the system

$$X' = \begin{bmatrix} 2 & 4 & 4 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{bmatrix} X$$

Question 46/352: Solve the IVP

$$X' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$