8.2 Homogeneous Linear Systems

Objectives:

- To study the relation between eigenvalues (& eigenvectors) of the matrix A with solutions of homogeneous linear system $X' = AX$
- To solve homogeneous systems $X' = AX$ if
  - A has distinct real eigenvalues
  - A has distinct complex eigenvalues
  - A has both distinct real and complex eigenvalues

Observation: **Relation of eigenvalues (& eigenvectors) with solutions of homogeneous systems**

- Consider a homogeneous linear system

$$X' = AX \qquad (*)$$

- Question: When a vector of the form $X = \mathbf{v}e^{\lambda t}$ will be a solution of (*)?
- Answer: $X = \mathbf{v}e^{\lambda t} \;\Rightarrow\; X' = \mathbf{v}\lambda e^{\lambda t}$

  Using X, X' in (*) we see that

  $\mathbf{v}\lambda e^{\lambda t} = A\mathbf{v}e^{\lambda t} \;\Leftrightarrow\; \mathbf{v}\lambda = A\mathbf{v}$ or

  $A\mathbf{v} = \lambda\mathbf{v}$

$\boxed{X = \mathbf{v}e^{\lambda t} \text{ is a solution}}$ of (*)

$$\Updownarrow$$

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Leftrightarrow \quad (A - \lambda I)\mathbf{v} = 0$$

(i.e. $\boxed{\lambda \text{ is an eigenvalue of } A}$ and $\boxed{\mathbf{v} \text{ is an eigenvector of } A}$)

By knowing eigenvalues & eigenvectors we can find solutions of (*)

We will study solutions of homogeneous systems as

- distinct real eigenvalues of A {**this section**}
- distinct complex eigenvalues of A {**this section**}
- Repeated eigenvalues of A {**next section**}

Solving homogeneous linear systems

Case I: Distinct real eigenvalues

- Given $X' = AX \qquad (*) \quad (A: n \times n)$

  and A has n real distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

  - We need n independent solutions
  - Each eigenvalue will give one.

❖ For each eigenvalue $\boxed{\lambda_i}$, find an associated eigenvector $\boxed{\mathbf{v}_i}$

➢ The solution vector associated to this eigenvalue is $\boxed{X_i = \mathbf{v}_i e^{\lambda_i t}}$

❖ Find all solution vectors and write general solution as

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

or $$X = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n \mathbf{v}_n e^{\lambda_n t}$$

Question 8/351: Find general solution of the system

$$\frac{dx}{dt} = 2x - y$$

$$\frac{dy}{dt} = 5x + 10y + 4z$$

$$\frac{dz}{dt} = 5y + 2z$$

Question 14/351: Solve the IVP

$$X' = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} X, \qquad X(0) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Solving homogeneous linear systems
Case II: Repeated eigenvalues

- Given $X' = AX \qquad (*)$

  and A has repeated eigenvalues.
- In such cases it may not always be possible to find linearly independent solutions of the homogeneous systems.
- If m is a positive integer and $(\lambda - \lambda_1)^m$ is a factor of the characteristic equation while $(\lambda - \lambda_1)^{m+1}$ is not a factor, then λ_1 is said to be an ***eigenvalue of multiplicity*** of m .

- In general, for some $n \times n$ matrices A it may be possible to find m linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_m$ corresponding to an eigenvalue λ_1 of multiplicity $m \le n$. In this case the general solution of the system contains the linear combination

$$c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_1 t} + \cdots + c_m \mathbf{v}_m e^{\lambda_1 t}$$

- If there is only one eigenvector corresponding to the eigenvalue λ_1 of multiplicity m , then m linearly independent solutions of the form

$$X_1 = \mathbf{v}_{11} e^{\lambda_1 t}$$

$$X_2 = \mathbf{v}_{21} t e^{\lambda_1 t} + \mathbf{v}_{22} e^{\lambda_1 t}$$

$$\vdots$$

$$X \;= \mathbf{v}_{m1} \frac{t^{m-1}}{(m-1)!} e^{\lambda_1 t} + \mathbf{v}_{m2} \frac{t^{m-2}}{(m-2)!} e^{\lambda_1 t} + \cdots + \mathbf{v}_{mm} e^{\lambda_1 t}$$

Where $\mathbf{v}_{ij}$ are column vectors, can always be found.

Procedure for Finding Linearly Independent Vectors when multiplicity of Eigenvalues is > 1

We consider a 3×3 matrix.

Step 1: Assume that matrix has 3 eigenvalues $\lambda_1, \lambda_1, \lambda_2$.

Step 2: Eigenvalues λ_1 has multiplicity "2" while λ_2 is unique.

Step 3: Two cases for λ_1

λ_1 has zero defect
(that is, for one eigenvalue we get two linearly independent eigenvectors. Thus defect is: 2-2=0

In this case the matrix admits two linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$
Solution of the system for this eigenvalue is

$$X_1 = \mathbf{v}_1 e^{\lambda_1 t} \text{ and } X_2 = \mathbf{v}_2 e^{\lambda_1 t}$$

λ_1 has defect
(that is, for one eigenvalue we get one linearly independent eigenvector. Thus defect is: 2-1=1

In this case the matrix admits two linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$
Solution of the system for this eigenvalue is

$$X_1 = \mathbf{v}_1 e^{\lambda_1 t} \text{ and } X_2 = (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda_1 t}$$

Example with Defect 1

Solve the system

$$x' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} x$$

Step1 Eigenvalues

$$\begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 6\lambda + 9 = 0 \Rightarrow \lambda = -3, -3$$

Repeated Eigenvalue of multiplicity 2

Step2 Eigenvectors with respect above Eigenvalue

- Use $\lambda_1 = -3$ in the matrix $(A - \lambda_1 I)$ and reduce it to Echelon form:

$$\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \xrightarrow{R_1/6} \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \xrightarrow[R_2 - 2R_1]{} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

- The Eigenvector is: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3r \\ r \end{pmatrix} = 3r \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

- One Solution is: $\vec{x} = v_1 e^{\lambda_1 t} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}$

No Second Eigenvector Exists.
Thus there is a defect: **2-1=1**

Step2 Procedure for **finding Second** linearly independent **Eigenvectors**

- Require that
  - $(A - \lambda I) v_2 = v_1$

    $\Rightarrow (A - \lambda I)^2 v_2 = (A - \lambda I) v_1 \qquad (1)$

    $Use\, (A - \lambda I) v_1 = 0$ *above to get*

    $(A - \lambda I)^2 v_2 = 0$

  - $(A - \lambda I) v_1 = 0 \qquad (2)$
- Solve $(A - \lambda I) v_2 = v_1$ by reducing it to Echelon form

Summary
One Defect Case

- $(A - \lambda I) v_2 = v_1$
- $(A - \lambda I) v_1 = 0$

• Let $v_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. Then solving above equation is equivalent to reducing

• $\left(\begin{array}{cc|c} 6 & -18 & 3 \\ 2 & -6 & 1 \end{array}\right)$ to Echelon form.

$$\left(\begin{array}{cc|c} 6 & -18 & 3 \\ 2 & -6 & 1 \end{array}\right) \xrightarrow{R_1/6} \left(\begin{array}{cc|c} 1 & -3 & 1/2 \\ 2 & -6 & 1 \end{array}\right) \xrightarrow[R_2-2R_1]{} \left(\begin{array}{cc|c} 1 & -3 & 1/2 \\ 0 & 0 & 0 \end{array}\right)$$

• The Corresponding Eigenvector is:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3r \\ r \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = r\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

$Set\, r = 0$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

Step3 Second Solution is:

$$x_2 = (v_1 t + v_2)e^{\lambda t} = \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}\right] e^{-3t}$$

Step4 General Solution is:

$$\vec{x} = \left\{ c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}\right]\right\} e^{-3t}$$

Example with Defect zero

Solve the system $x' = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} x$

Step1: Eigenvalues

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0 \Rightarrow -3\begin{vmatrix} 2 & 1-\lambda \\ -1 & -2 \end{vmatrix} + 6\begin{vmatrix} -2-\lambda & 2 \\ -1 & -2 \end{vmatrix} - \lambda\begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$-3[-4+1-\lambda]+6[4+2\lambda+2]-\lambda\left[\lambda^2+\lambda-2-4\right]=0$$

$$3\lambda+9+12\lambda+36-\lambda^3-\lambda^2+6\lambda=0$$

$$\lambda^3+\lambda^2-21\lambda-45=0$$

$$\lambda_{1,2}=-3,-3,\ \lambda_3=5$$

Step2: Eigenvectors

- Use $\lambda_1=-3$ in $(A-\lambda_1 I)$ and reduce it to Echelon form:

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2r+3s \\ r \\ s \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -r\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix},\quad v_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

Corresponding to Eigen value of multiplicity 2, we get two linearly independent vectors. Hence **defect is zero**

- Use $\lambda_1=5$ in $(A-\lambda_1 I)$ and reduce it to Echelon form:

$$\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix} \xrightarrow[R_1\leftrightarrow R_3]{} \begin{pmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{pmatrix} \xrightarrow{-1\times R_1} \begin{pmatrix} 1 & 2 & 5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{pmatrix}$$

$$\xrightarrow[R_3+7R_1]{R_2-2R_1} \begin{pmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{pmatrix} \xrightarrow[R_3/16]{R_2/-8} \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[R_3-R_2]{R_1-2R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -r \\ -2r \\ r \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -r\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Step3 Solutions

Substitute the three vectors in $x_i = v_i e^{\lambda_i t}$ to get three solutions:

- $$x_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} e^{-3t}$$

- $$x_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} e^{-3t}$$

- $$x_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} e^{5t}$$

Step4 General Solution

$$x = c_1 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} e^{-3t} + c_3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} e^{5t}$$

Procedure for finding two missing Eigenvector when Eigenvalue has multiplicity 3

Two Defect Case

- **Finding Eigenvectors given v_1**
  - $(A-\lambda I)v_3 = v_2$
  - $(A-\lambda I)v_2 = v_1$
  - $(A-\lambda I)v_1 = 0$

- **Three Corresponding Solutions**
  - $x_1 = v_1 e^{\lambda t}$
  - $x_2 = (v_1 t + v_2)e^{\lambda t}$
  - $x_3 = \left(\dfrac{v_1 t^2}{2} + v_2 t + v_3\right)e^{\lambda t}$

Example: Solve the system $x' = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} x$

Step1: Eigenvalues

$$\begin{vmatrix} 2-\lambda & 1 & 6 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_{1,2,3} = 2$$

Step2: Eigenvectors (Reduce $(A-\lambda_1 I)$ and reduce it to Echelon form)

$$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2/2} \begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - 6R_2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We need three Eigen vectors but get only 1. The defect is 3-1=2. To find two vectors we use $(A-\lambda I)v_3 = v_2$ and $(A-\lambda I)v_2 = v_1$ such that $(A-\lambda I)v_1 = 0$

• **Finding** v_2

Reduce to Echelon form the matrix ($(A-\lambda I)v_2 = v_1$:

$$\left(\begin{array}{ccc|c} 0 & 1 & 6 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2/5} \left(\begin{array}{ccc|c} 0 & 1 & 6 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1-6R_2} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \\ 1 \\ 0 \end{pmatrix} = r\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} setting\ r = 0$$

• **Finding** v_3 Reduce to Echelon form the matrix $(A-\lambda I)v_3 = v_2$:

$$\left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2/5} \left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 1/5 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1-6R_2} \left(\begin{array}{ccc|c} 0 & 1 & 0 & -6/5 \\ 0 & 0 & 1 & 1/5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \\ -6/5 \\ 1/5 \end{pmatrix} = r\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix} setting\ r = 0$$

Step3: Solutions

• $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$

• $x_2 = (v_1 t + v_2)e^{2t} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] e^{2t}$

• $x_3 = (\frac{v_1 t^2}{2} + v_2 t + v_3) = \left[\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix}\right] e^{2t}$

Step4: General Solution

It is given by

$$x = c_1 x_1 + c_2 x_2 + c_3 x_3$$

Exercise 8.2.2

Q19 Find general solution of $x' = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} x$

Step1: Eigenvalues

$$\begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = 0 \Rightarrow \lambda = 0,0$$

Step2: Eigenvectors corresponding to Eigenvalue "0"

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \xrightarrow{R_1/3} \begin{pmatrix} 1 & -1/3 \\ 9 & -3 \end{pmatrix} \xrightarrow{R_2 - 9R_1} \begin{pmatrix} 1 & -1/3 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r/3 \\ r \end{pmatrix} = r/3 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Deficit = 2-1=1. Hence use $(A - \lambda I)v_2 = v_1$

- **Second Eigenvectors using** $(A - \lambda I)v_2 = v_1$

Reduce $\left(\begin{array}{cc|c} 3 & -1 & 1 \\ 9 & -3 & 3 \end{array}\right)$ **to Echelon form:**

$$\left(\begin{array}{cc|c} 3 & -1 & 1 \\ 9 & -3 & 3 \end{array}\right) \xrightarrow{R_1/3} \left(\begin{array}{cc|c} 1 & -1/3 & 1/3 \\ 9 & -3 & 3 \end{array}\right) \xrightarrow{R_2 - 9R_1} \left(\begin{array}{cc|c} 1 & -1/3 & 1/3 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r/3 \\ r \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$$

Step3: Solutions

$$x_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$x_2 = (v_1 t + v_2)e^{\lambda t} = \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}\right]$$

Step4: General Solution

$$x = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}\right]$$

Q25 Find general solution of $x'=\begin{pmatrix}5 & -4 & 0\\ 1 & 0 & 2\\ 0 & 2 & 5\end{pmatrix}x$

Step1: Eigenvalues

$$\begin{vmatrix}5-\lambda & -4 & 0\\ 1 & -\lambda & 2\\ 0 & 2 & 5-\lambda\end{vmatrix}=0 \Rightarrow \lambda = 5,5,0$$

Step2: Eigenvector when $\lambda = 5$

$$\begin{pmatrix}0 & -4 & 0\\ 1 & -5 & 2\\ 0 & 2 & 0\end{pmatrix}\xrightarrow{R_1\leftrightarrow R_2}\begin{pmatrix}1 & -5 & 2\\ 0 & -4 & 0\\ 0 & 2 & 0\end{pmatrix}\xrightarrow[R_3/2]{R_2/-4}\begin{pmatrix}1 & -5 & 2\\ 0 & 1 & 0\\ 0 & 1 & 0\end{pmatrix}$$

$$\xrightarrow[R_3-R_2]{R_1+5R_2}\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\Rightarrow\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}-2r\\ 0\\ r\end{pmatrix}=-r\begin{pmatrix}2\\ 0\\ -1\end{pmatrix}\Rightarrow v_1=\begin{pmatrix}2\\ 0\\ -1\end{pmatrix}$$

We **need** two Eigen vectors but get only 1. This implies that the defect is 2-1=1. To find second vector we use and $(A-\lambda I)v_2 = v_1$ such that $(A-\lambda I)v_1 = 0$

- **Finding Second Eigenvector when** $\lambda = 5$ **(use** $(A-\lambda I)v_2 = v_1$)

$$\left(\begin{array}{ccc|c}0 & -4 & 0 & 2\\ 1 & -5 & 2 & 0\\ 0 & 2 & 0 & -1\end{array}\right)\xrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{ccc|c}1 & -5 & 2 & 0\\ 0 & -4 & 0 & 2\\ 0 & 2 & 0 & -1\end{array}\right)\xrightarrow[R_3/2]{R_2/-4}\left(\begin{array}{ccc|c}1 & -5 & 2 & 0\\ 0 & 1 & 0 & -1/2\\ 0 & 1 & 0 & -1/2\end{array}\right)$$

$$\xrightarrow[R_3-R_2]{R_1+5R_2}\left(\begin{array}{ccc|c}1 & 0 & 2 & -5/2\\ 0 & 1 & 0 & -1/2\\ 0 & 0 & 0 & 0\end{array}\right)\Rightarrow\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}-2r-5/2\\ -1/2\\ r\end{pmatrix}=-r\begin{pmatrix}2\\ 0\\ -1\end{pmatrix}+\begin{pmatrix}-5/2\\ -1/2\\ 0\end{pmatrix}\Rightarrow$$

$$v_2=\begin{pmatrix}-5/2\\ -1/2\\ 0\end{pmatrix}$$

- **Finding Second Eigenvector when** $\lambda = 0$

$$\begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 2 \\ 5 & -4 & 0 \\ 0 & 2 & 5 \end{pmatrix} \xrightarrow{R_2 - 5R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -4 & -10 \\ 0 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow{R_2 \times -1/4} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5/2 \\ 0 & 2 & 5 \end{pmatrix} \xrightarrow[R_3 - 2R_2]{} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2r \\ -5/2r \\ r \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -r/2 \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix}$$

Step3: Solutions

$$x_1 = v_1 e^{5t} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} e^{5t}$$

$$x_2 = [v_1 t + v_2] e^{5t} = \left[\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} t + \begin{pmatrix} -5/2 \\ -1/2 \\ 0 \end{pmatrix} \right] e^{5t}$$

$$x_3 = v_3 e^{0t} = \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix}$$

Step4: General Solution

$$x_1 = \left\{ c_1 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + c_2 \left[\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} t + \begin{pmatrix} -5/2 \\ -1/2 \\ 0 \end{pmatrix} \right] \right\} e^{5t} + c_3 \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix}$$

A word on Check that your Vectors and solutions are correct
The following must be satisfied

(Matrix of Eigenvector)$^{-1}$(Given matrix)(Matrix of Eigenvectors)= $\begin{pmatrix} 5\vdots & \ldots 1\vdots & 0\vdots \\ 0 & \ldots 5\vdots & 0\vdots \\ 0 & \ldots 0\vdots & 0\vdots \end{pmatrix}$

Q29 Solve the IVP $x' = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} x$ **with** $x(0) = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

Step1: Eigenvalues

$$\begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 8\lambda + 16 = 0 \Rightarrow \lambda = 4,4$$

Step2: Eigenvectors

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow -R_2} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \xrightarrow[R_2+2R_1]{} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2r \\ r \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = r\begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We **need** two Eigen vectors but get only 1. This implies that the defect is 2-1=1. To find second vector we use and $(A - \lambda I)v_2 = v_1$ such that $(A - \lambda I)v_1 = 0$

• **Finding Second Eigenvector (solve** $(A - \lambda I)v_2 = v_1$)

$$\left(\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow -R_2} \left(\begin{array}{cc|c} 1 & -2 & -1 \\ -2 & 4 & 2 \end{array}\right) \xrightarrow[R_2+2R_1]{} \left(\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2r-1 \\ r \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = r\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Step3: Solutions

$$x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t}$$

$$x_2 = (v_1 t + v_2)e^{4t} = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right] e^{4t}$$

Step4: General Solution

$$x = \left\{ c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right] \right\} e^{4t}$$

Step5: Initial Condition $x(0)=\begin{pmatrix}-1\\6\end{pmatrix}$

Apply above condition in general solution to get:

$$\begin{pmatrix}-1\\6\end{pmatrix}=\left\{c_1\begin{pmatrix}2\\1\end{pmatrix}+c_2\left[\begin{pmatrix}-1\\0\end{pmatrix}\right]\right\}$$

To determine constants reduce to Echelon form the following matrix

$$\left(\begin{array}{cc|c}2 & -1 & -1\\1 & 0 & 6\end{array}\right)\xrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{cc|c}1 & 0 & 6\\2 & -1 & -1\end{array}\right)\xrightarrow[R_2-2R_1]{}\left(\begin{array}{cc|c}1 & 0 & 6\\0 & -1 & -13\end{array}\right)$$

$$\xrightarrow[R_2\times -1]{}\left(\begin{array}{cc|c}1 & 0 & 6\\0 & 1 & 13\end{array}\right)\Rightarrow c_1=6,\quad c_2=13$$

The solution of the IVP is:

$$x=\left\{6\begin{pmatrix}2\\1\end{pmatrix}+13\left[\begin{pmatrix}2\\1\end{pmatrix}t+\begin{pmatrix}-1\\0\end{pmatrix}\right]\right\}e^{4t}$$

Check that our answer is correct

```
Inverse[((2 -1
          1  0))].((2 4
                    -1 6)).(2 -1
                            1  0) // MatrixForm
```

$$\begin{pmatrix}4 & 1\\0 & 4\end{pmatrix}$$

Solving homogeneous linear systems

Case III: Distinct complex eigenvalues

- Given $X' = AX$ (*)

  and A has distinct complex eigenvalues
- Recall: complex eigenvalues and eigenvectors occur in conjugate pairs.

❖ For each pair $\boxed{p \pm iq}$ of eigenvalues, find the associated pair of eigenvectors and split into real and imaginary parts as $\boxed{\mathbf{a} \pm i\mathbf{b}}$

➢ The solution vectors associated to the pair of eigenvalues $p \pm iq$ are

$$\boxed{\begin{aligned} X_1 &= e^{pt}\left(\mathbf{a}\cos qt - \mathbf{b}\sin qt\right) \\ X_2 &= e^{pt}\left(\mathbf{b}\cos qt + \mathbf{a}\sin qt\right) \end{aligned}}$$

❖ Find all solution vectors and write general solution.

Question 40/352: Find the general solution of the system

$$\frac{dx}{dt} = 2x + y + 2z$$

$$\frac{dy}{dt} = 3x + 6z$$

$$\frac{dz}{dt} = -4x - 3z$$

Solving homogeneous linear systems

Mixture of Case I and Case III

- Given $X' = AX \qquad (*)$

  and A has distinct real and complex eigenvalues

❖ For each real eigenvalue $\boxed{\lambda_i}$, find eigenvector $\boxed{\mathbf{v}_i}$

➢ Then $\boxed{X_i = \mathbf{v}_i e^{\lambda_i t}}$ is corresponding term in solution

❖ For each complex pair $\boxed{p \pm iq}$ of eigenvalues, find the pair of eigenvectors $\boxed{\mathbf{a} \pm i\mathbf{b}}$

➢ Then

$$\boxed{\begin{array}{l} e^{pt}\left(\mathbf{a}\cos qt - \mathbf{b}\sin qt\right) \\ e^{pt}\left(\mathbf{b}\cos qt + \mathbf{a}\sin qt\right) \end{array}}$$

are corresponding terms in the general solution

Question 44/352: Find the general solution of the system

$$X' = \begin{bmatrix} 2 & 4 & 4 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{bmatrix} X$$

Question 46/352: Solve the IVP

$$X' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} X, \qquad X(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$