8.2 Homogeneous Linear Systems

Objectives:

- To study the relation between eigenvalues (& eigenvectors) of the matrix A with solutions of homogeneous linear system $X' = AX$
- To solve homogeneous systems $X' = AX$ if
  - A has distinct real eigenvalues
  - A has repeated eigenvalues
  - A has distinct complex eigenvalues
  - A has both distinct real and complex eigenvalues

Observation: **Relation of eigenvalues (& eigenvectors) with solutions of homogeneous systems**

- Consider a homogeneous linear system

$$X' = AX \qquad (*)$$

- Question: When a vector of the form $X = \mathbf{v}e^{\lambda t}$ will be a solution of (*)?
- Answer: $X = \mathbf{v}e^{\lambda t} \Rightarrow X' = \mathbf{v}\lambda e^{\lambda t}$

  Using X, X' in (*) we see that

  $\mathbf{v}\lambda e^{\lambda t} = A\mathbf{v}e^{\lambda t} \Leftrightarrow \mathbf{v}\lambda = A\mathbf{v}$ or

  $A\mathbf{v} = \lambda \mathbf{v}$

$X = \mathbf{v}e^{\lambda t}$ is a solution of (*)

$$\Updownarrow$$

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Leftrightarrow \quad (A - \lambda I)\mathbf{v} = 0$$

(i.e. λ is an eigenvalue of A and $\mathbf{v}$ is an eigenvector of A)

By knowing eigenvalues & eigenvectors we can find solutions of (*)

We will study solutions of homogeneous systems as

- distinct real eigenvalues of A
- Repeated eigenvalues of A
- distinct complex eigenvalues of A

Solving homogeneous linear systems

Case I: Distinct real eigenvalues

- Given $X' = AX$ (*) $(A: n \times n)$

and A has n real distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

- We need n independent solutions
- Each eigenvalue will give one.

❖ For each eigenvalue λ_i, find an associated eigenvector $\mathbf{v}_i$

➢ The solution vector associated to this eigenvalue is $X_i = \mathbf{v}_i e^{\lambda_i t}$

❖ Find all solution vectors and write general solution as

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

or $X = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n \mathbf{v}_n e^{\lambda_n t}$

Question 8/351: Find general solution of the system

$$\frac{dx}{dt} = 2x - 7y$$

$$\frac{dy}{dt} = 5x + 10y + 4z$$

$$\frac{dz}{dt} = 5y + 2z$$

Question 14/351: Solve the IVP

$$X' = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} X, \qquad X(0) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Solving homogeneous linear systems

Case II: Repeated eigenvalues

When repeated eigenvalues with multiplicity k have k independent eigenvectors

- Given $X' = AX$
- If λ_i is an eigenvalue (of A) of multiplicity k with k independent eigenvectors, then
  - find k independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$
  - the solution terms associated to the eigenvalue λ_i are

$$c_1 \mathbf{v}_1 e^{\lambda_i t} + c_2 \mathbf{v}_2 e^{\lambda_i t} + \cdots + c_k \mathbf{v}_k e^{\lambda_i t}$$

Question 24/351: Find the general solution of the system

$$\frac{dx}{dt} = 3x + 2y + 4z$$

$$\frac{dy}{dt} = 2x + 2z$$

$$\frac{dz}{dt} = 4x + 2y + 3z$$

Solving homogeneous linear systems

{when repeated eigenvalues of multiplicity *2* have *one* independent eigenvetor}

- Given $X' = AX$
- If $\boxed{\lambda_i}$ is an eigenvalue (of A) of multiplicity 2 with one independent eigenvector $\mathbf{v}_1$,
  - find a generalized eigenvector $\boxed{\mathbf{v}_2}$ for λ_i

    by solving $$\boxed{(A-\lambda I)\mathbf{v}_2 = \mathbf{v}_1}$$
  - Then the solution terms associated to the eigenvalue $\boxed{\lambda_i}$ are

$$\boxed{c_1\mathbf{v}_1 e^{\lambda_i t} + c_2(\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda_i t}}$$

Solving homogeneous linear systems

{when repeated eigenvalues of multiplicity *3*

have *one* independent eigenvetor}

- Given $X' = AX$
- If λ_i is an eigenvalue (of A) of multiplicity 3 with *one* independent eigenvector $\mathbf{v}_1$,
    - find two generalized eigenvectors $\mathbf{v}_2, \ \mathbf{v}_3$ for λ_i by solving $(A-\lambda I)\mathbf{v}_2 = \mathbf{v}_1$ and $(A-\lambda I)\mathbf{v}_3 = \mathbf{v}_2$
    - Then the solution terms associated to the eigenvalue λ_i are

$$c_1\mathbf{v}_1 e^{\lambda_i t} + c_2(\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda_i t} + c_3\left(\mathbf{v}_1\frac{t^2}{2} + \mathbf{v}_2 t + \mathbf{v}_3\right)e^{\lambda_i t}$$

Question 28/352: Find the general solution of the system

$$X' = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} X$$

Solving homogeneous linear systems

Case III: Distinct complex eigenvalues

- Given $X' = AX$ (*)

and A has distinct complex eigenvalues

- Recall: complex eigenvalues and eigenvectors occur in conjugate pairs.

❖ For each pair $\boxed{p \pm iq}$ of eigenvalues, find the associated pair of eigenvectors and split into real and imaginary parts as $\boxed{\mathbf{a} \pm i\mathbf{b}}$

➢ The solution vectors associated to the pair of eigenvalues $p \pm iq$ are

$$\boxed{\begin{aligned} X_1 &= e^{pt}\left(\mathbf{a}\cos qt - \mathbf{b}\sin qt\right) \\ X_2 &= e^{pt}\left(\mathbf{b}\cos qt + \mathbf{a}\sin qt\right) \end{aligned}}$$

❖ Find all solution vectors and write general solution.

Question 46/352: Solve the IVP

$$X' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} X, \qquad X(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

Solving homogeneous linear systems

Mixture of Case I and Case III

- Given $X' = AX \qquad (*)$

and A has distinct real and complex eigenvalues

- For each real eigenvalue λ_i, find eigenvector $\mathbf{v}_i$
  - Then $X_i = \mathbf{v}_i e^{\lambda_i t}$ is corresponding term in solution

- For each complex pair $p \pm iq$ of eigenvalues, find the pair of eigenvectors $\mathbf{a} \pm i\mathbf{b}$
  - Then

$$e^{pt}\left(\mathbf{a}\cos qt - \mathbf{b}\sin qt\right)$$
$$e^{pt}\left(\mathbf{b}\cos qt + \mathbf{a}\sin qt\right)$$

are corresponding terms in the general solution

Question 40/352: Find the general solution of the system

$$\frac{dx}{dt} = 2x + y + 2z$$

$$\frac{dy}{dt} = 3x + 6z$$

$$\frac{dz}{dt} = -4x - 3z$$