8.1 Preliminary Theory (Linear Systems)

Objectives:

- To learn how to write a system of 1^{st} order linear differential equations in matrix form
- To know basic facts about solutions of homogeneous linear systems
- To learn how to check linear independence of solutions of homogeneous systems

Standard form of linear systems

- Linear System: A system of ODE's in which unknowns appear linearly.
- Examples:

1. $$x_1' = 3x_1 - 5x_2 + 2e^t$$ $$x_2' = 2x_1 + x_2 - \sin t$$ Linear

2. $$x_1' = 3x_1x_2 - 5x_2$$ $$x_2' = 2x_1 + x_2$$ Non Linear

3. $$x_1' = -2x_2$$ $$x_2' = 3(x_1)^2 + 4$$ Non Linear

Standard form of a 1st order linear system

$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots\cdots + p_{1n}(t)x_n + f_1(t)\\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots\cdots + p_{2n}(t)x_n + f_2(t)\\
&\vdots\\
&\vdots\\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots\cdots + p_{nn}(t)x_n + f_n(t)
\end{aligned} \qquad (*)$$

- If all $f_i(t) = 0$ then homogeneous system
- Otherwise non-homogeneous

Matrix form of a linear system

- The above linear system (*) can be written in matrix form as

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & & p_{2n}(t) \\ \vdots & \vdots & & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

- If we set $X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ then we have

$$\frac{dX}{dt} = P(t) \cdot X + f(t)$$

or

$$X' = PX + f \qquad \textbf{(**)}$$

Question 6/336: Write the linear system

$$\frac{dx}{dt} = -3x + 4y + e^{-t} \sin 2t$$
$$\frac{dy}{dt} = 5x + 9z + 4e^{-t} \cos 2t$$
$$\frac{dz}{dt} = y + 6z - e^{-t}$$

in matrix form.

Meaning of solution of a homogeneous system

- Given a homogeneous linear system

$$X' = AX \qquad (*) \qquad \text{where } X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

- **Solution:** Functions $x_1, x_2, \cdots, x_n$ or a vector $X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

which satisfies equation (*).

Question 16/337: Verify that the vector $X = \begin{bmatrix} \sin t \\ -\frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\sin t + \cos t \end{bmatrix}$

is a solution of $X' = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} X$.

Facts about solutions of homogeneous linear systems

Given a homogeneous linear system of *n* equations

$$X' = AX \qquad (*)$$

If vectors $X_1, X_2, \cdots, X_k$ are solutions of (*) then $c_1X_1 + c_2X_2 + \cdots + c_kX_k$ is also a solution

- Linear combination of solutions is a solution
- Can generate infinitely many solutions from few known solutions

How many solutions do we need?

(to know all solutions)

If $X_1, X_2, \cdots, X_n$ are solutions of (*) then the general solution of (*) is

$$X = c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

- All solutions can be obtained from *n* independent solutions
- How to check independence {see next}

How to check linear independence of solutions

Given a homogeneous linear system of n equations

$$X' = AX \qquad (*)$$

and solution vectors $X_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}, X_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}, \cdots, X_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$

The solutions $X_1, X_2, \cdots, X_n$ are linearly independent

$$\Leftrightarrow$$

$$W(X_1 \quad X_2 \quad \cdots \quad X_n) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \neq 0$$

for every t in the interval.

The function $W(t)$ is called Wronskian of solution vectors

General Solution – Nonhomogeneous Systems: Let X_p be a Given solution of the nonhomogeneous system $X' = AX + F$ on an interval I, and let

$$X = c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

denote the general solution on the same interval of the associated homogeneous system $X' = AX$. Then the ***general solution*** of the nonhomogeneous system $X' = AX + F$ is $X = X_c + X_p$.

The general solution X_c of the associated homogeneous system is called the ***complementary function*** of nonhomogeneous system.

Question 24/337: Verify that $X_p = \begin{bmatrix} \sin 3t \\ 0 \\ \cos 3t \end{bmatrix}$ is a particular solution of

the system $X' = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ -6 & 1 & 0 \end{bmatrix} X + \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \sin 3t$.