6.2 Solutions About Singular Points

Objectives:

- Define regular and irregular singular points
- The solution of ODEs which have singular points.
- To study the Frobenius method to find a series solutions of a second order ODE about regular singular points

Regular and Irregular Singular Points:

- Given $y'' + P(x)y' + Q(x)y = 0$ (*)
- Given a singular point x_0. (Recall, a point x_0 is called a ***singular point*** of ODE (*) if at least one of the functions $P(x)$ and $Q(x)$ is not analytic at x_0.)

- x_0 **is regular singular point** if $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are both analytic at x_0
- x_0 **is irregular singular point** if one or both of the functions $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are not analytic at x_0

Example: Is $x=1$ a regular singular point of

$$(x-1)^2 y'' + 2x(x-1)y' + 3(x+1)y = 0\,?$$

Solution:

- Writing as $y'' + \dfrac{2x(x-1)}{(x-1)^2}y' + \dfrac{3(x+1)}{(x-1)^2}y = 0$, we see that
- $P(x) = \dfrac{2x}{x-1}$ and $Q(x) = \dfrac{3(x+1)}{(x-1)^2}$
- Hence $(x-1)P(x) = 2x$ is analytic at $x=1$
- and $(x-1)^2 Q(x) = 3(x+1)$ is analytic at $x=1$

Hence $x=1$ is regular singular

Example: Is $x=0$ a regular singular point of $x^2 y'' + 2(x-1)y' + xy = 0$?

Solution:

- Writing as $y'' + \dfrac{2(x-1)}{x^2}y' + \dfrac{1}{x}y = 0$, we see that
- $P(x) = \dfrac{2(x-1)}{x^2}$ and $Q(x) = \dfrac{1}{x}$
- Since $xP(x) = \dfrac{2(x-1)}{x}$ is not analytic at $x=0$, it is not a regular singular point.

Question 2/257: Determine singular points and classify as regular or irregular singular points of the ODE $x(x+3)^2 y'' - y = 0$.

Question 7/257: Determine singular points and classify as regular or irregular singular points of the ODE

$(x^2 + x - 6)y'' + (x+3)y' + (x-2)y = 0$.

Frobenius Theorem:

- Given $y'' + P(x)y' + Q(x)y = 0$ (*)
- If $x = x_0$ is a regular singular point then
  - we can always find at least one power series solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

Converges in an interval around the point $x = x_0$

- Here r is a number to be determined
- See Frobenius' method below to see how to find r

Our strategy

- We have a guarantee of at least one solution.
- We will find one solution by Frobenius method and then use this solution to find the other solution using reduction of order (learnt in 4.2)

Preparation for Frobenius' method

Practice of reduction of order

- We see from above that we would need to use reduction of order to find the second solution.

First we do practice of

- using reduction of order to find second solution using a given series solution of a second order linear equation

Some tricks to handle series will be needed.

Recall Reduction of Order

- Given $y'' + P(x)y' + Q(x)y = 0$
- Given a solution $y_1(x)$
- Then a second solution $y_2(x)$ is given by

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx$$

Method of Frobenius to find series solution near a regular singularpoint

- Given $y'' + P(x)y' + Q(x)y = 0$ (*)
- Choose the regular singular point to center the series solution.
- Consider the solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$
- Take necessary derivatives and put in (*)
- Shift indices to write the equation as one series
- Compare coefficients of smallest power of x.
    - This gives an equation (of degree ≤ 2) in r called indicial equation. Find roots of indicial equation and **take r as the largest root**.
- Compare other coefficients and use the value of r to get the recurrence relation
- Use recurrence relation to find all c_n's.

  These give one solution.
- Find other solution by reduction of order.

Roots of Indicial Equation and Solutions of the ODE with Regular Singular Point at $x_0 = 0$

- **Distinct Roots** of indicial Equation r_1, r_2 such that $r_1 - r_2$ is **not +ve integer**
  - **Solutions are**:

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \; c_o \neq 0,$$

$$y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}, \; b_o \neq 0,$$

- **Distinct Roots** of indicial Equation r_1, r_2 such that $r_1 - r_2$ is **+ve integer**
  - **Solutions are**:

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \; c_o \neq 0,$$

$$y_2 = c\, y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \; b_o \neq 0,$$

- **Equal Roots** of indicial Equation r_1, r_1
  - **Solutions are**:

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \; c_o \neq 0,$$

$$y_2 = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1}$$

Exercise 6.2

Q1 **Determine singular points of the ODE** $x^3 y''+4x^2 y'+3y = 0$

- **Step1**: **Re-write ODE**

$$y''+\frac{4}{x}y'+\frac{3}{x^3}y = 0$$

- **Step2** **Identify Singular points**

$x_o = 0$ is a singular point of ODE

- **Step3** **Classification of Singular point**

- $x_o = 0$

$$(x)\left[\frac{1}{x}\right] = 1 \text{ Analytic at 0}$$

$$(x^2)\left[\frac{1}{x^3}\right] = \frac{1}{x} \text{ Not Analytic at 0}$$

$x_o = 0$ **is a an irregular singular point of the ODE. Hence series solution of Frobenius type does not exist.**

Q2 **Determine singular points of the ODE** $x(x+3)^2 y''-y = 0$

- **Step1**: **Re-write ODE**

$$y''-\frac{3}{x(x+3)^2}y = 0$$

- **Step2** **Identify Singular points**

$x_o = 0$ and $x_o = -3$ are singularities of ODE

- **Step3** **Classification of Singular point**

- (A) $x_o = 0$

$$(x^2)\left[\frac{1}{x(x+3)^2}\right] = \frac{x}{(x+3)^2}, \textit{Analytic}$$

$x_o = 0$ **is a regular singular point of the ODE. Hence series solution of Frobenius type exists.**

• (B) $x_o = -3$

$$(x+3)^2\left[\frac{1}{x(x+3)^2}\right] = \frac{1}{x} \text{ Analytic}$$

$x_o = -3$ is a regular singular point of the ODE. Hence series solution of Frobenius type exists.

Q7 Determine singular points of the ODE $(x^2 + x - 6)y'' + (x+3)y' + (x-2)y = 0$

• **Step1**: **Re-write ODE**

$$y'' + \frac{(x+3)}{(x^2+x-6)}y' + \frac{(x-2)}{(x^2+x-6)}y = 0 \text{ or}$$

$$y'' + \frac{1}{(x-2)}y' + \frac{1}{(x+3)}y = 0$$

• **Step2** **Identify Singular points**

$x_o = 2$ and $x_o = -3$ are singularities of ODE

• **Step3** **Classification of Singular point**

• (A) $x_o = 2$

$$(x-2)\left[\frac{1}{(x+2)}\right] = 1, Analytic$$

$$(x-2)^2\left[\frac{1}{(x+3)}\right] = \frac{(x-2)^2}{(x+3)}, Analytic$$

$x_o = 2$ is a regular singular point of the ODE. Hence series solution of Frobenius type exists.

- (B) $x_o = -3$

$$(x+3)\left[\frac{1}{(x+2)}\right] = \frac{x+3}{x+2}, Analytic$$

$$(x+3)^2\left[\frac{1}{(x+3)}\right] = x+3, Analytic$$

$x = -3$ is a regular singular point of the ODE. Hence series solution of Frobenius type exists.

Q13 x=0 is a regular singularity of the ODE $x^2 y'' + (\frac{5}{3}x + x^2)y' - \frac{1}{3}y = 0$. With out solving discuss the number of series solutions we expect using Frobenius method.

- **Step1 Substitute in the ODE**
  - $y = \sum_{n=0}^{\infty} c_n x^{n+r}$
  - $y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$
  - $y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$
- **Step2 The ODE becomes**

$$\sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r} + \frac{5}{3}\sum_{n=0}^{\infty} c_n (n+r)x^{n+r} + \sum_{n=0}^{\infty} c_n (n+r)x^{n+r+1} - \frac{1}{3}\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Step3 Indicial Equation (*Expand above in lowest power of x i.e. x=0)* is:

$$(r)(r-1) + \frac{5}{3}r - \frac{1}{3} = 0$$

Step4 Difference of Roots

Roots are ***neither integers*** and ***nor differ by integer***. Two solutions may be found using Frobenius method**.**

Q15 x=0 is a regular singularity of the ODE $2xy''-y'+2y=0$. Show that the indicial roots of the singularity do not differ by an integer. Use Frobenius method to find two linearly independent series solutions about x=0.

Solution

- **Step1** **Substitute Series solution in the ODE**

  - $y=\sum_{n=0}^{\infty} c_n x^{n+r}$

  - $y'=\sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$

  - $y''=\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$

- **Step2** **Substitute above in the ODE to get**

$$x^r\left[2\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n-1}-\sum_{n=0}^{\infty} c_n(n+r)x^{n-1}+2\sum_{n=0}^{\infty} c_n x^n\right]=0 \quad (1)$$

- **Step3** **Indicial Equation**

$$x^r\left[\begin{array}{l}2c_o r(r-1)x^{-1}+2\sum_{n=1}^{\infty} c_n(n+r)(n+r-1)x^{n-1}-c_o r x^{-1}-\sum_{n=1}^{\infty} c_n(n+r)x^{n-1}\\ +2\sum_{n=0}^{\infty} c_n x^n\end{array}\right]=0$$

$$x^r\left[2c_o r(r-1)-c_o r\right]=0 \Rightarrow 2r^2-3r=0 \Rightarrow r=0, \quad r=3/2$$

Step4 **Recurrence Relation**

$$x^r\left[2\sum_{n=1}^{\infty} c_n(n+r)(n+r-1)x^{n-1} - \sum_{n=1}^{\infty} c_n(n+r)x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n\right] = 0$$

giving

$$x^r\left[2\sum_{k=0}^{\infty} c_{k+1}(k+r+1)(k+r)x^k - \sum_{k=0}^{\infty} c_{k+1}(k+r+1)x^k + 2\sum_{k=0}^{\infty} c_k x^k\right] = 0$$

giving

$$c_{k+1} = \frac{-2c_k}{(k+r+1)(2k+2r-1)} \text{ k=0, 1,2,...}$$

Step5 Substitute $r = 3/2$ above to get Recurrence Relation as

$$c_{k+1} = \frac{-2c_k}{(2k+5)(2k+2)} \text{ or } c_m = \frac{-2c_{m-1}}{(2m+3)m} \quad with\, k+1 = m = 1,2,\cdots$$

In question like this we can begin also by reducing the last term in (a) to summation over k-1 instead of changing summation from k-1 to k.

- **Working out Constants**

$$k = 0 \Rightarrow c_1 = \frac{-2c_o}{(5)(2)} = \frac{-c_o}{5}; \quad k = 1 \Rightarrow c_2 = \frac{-c_1}{14} = \frac{c_o}{70} \text{ etc}$$

Step6 Substitute $r = 0$ above to get Recurrence Relation as:

$$c_{k+1} = \frac{-2c_k}{(k+1)(2k-1)}$$

- **Working out Constants**

$$k = 0 \Rightarrow c_1 = \frac{-2c_o}{-1} = 2c_o; \quad k = 1 \Rightarrow c_2 = -c_1 = -2c_o \text{ etc}$$

Step7 General Solution

$$y = c_o + 2c_o x - 2c_o x^2 + \cdots + x^{3/2}(c_o - \frac{c_o}{5}x + \frac{c_o}{70}x^2 + \cdots)$$

$$= C(1 + 2x - 2x^2 + \cdots) + x^{3/2}E(1 - \frac{1}{5}x + \frac{1}{70}x^2 - \cdots)$$