6.1.2 Power Series Solutions

Objectives:

- Power Series Method to solve ODEs with polynomial and non polynomial coefficients
- Define Ordinary and Singular Points of an ODE
- Conditions under which the series solutions exist

Consider the linear second-order ODE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (1)$$

We first write the standard form of ODE by dividing by the leading coefficient $a_2(x)$, then we have

$$y'' + P(x)y' + Q(x)y = 0 \qquad (2)$$

Definitions:

- A point x_0 is called an ***ordinary point*** of ODE (1) if both $P(x)$ and $Q(x)$ are analytic at x_0 (A function is analytic at a point if it can be expressed in power series in $(x - x_0)$).
- A point x_0 is called an ***singular point*** of ODE (1) if at least one of the functions $P(x)$ and $Q(x)$ is not analytic at x_0.

- If $x = x_0$ is an ordinary point of (1), then we can always find two linearly independent solutions of the form of a power series centered at x_0, that is, $y = \sum_{n=0}^{\infty} c_n (x - x)^n$. The solution converges in $|x - x_0| < R$.

- A solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x)^n$ is said to be a ***solution about the ordinary point*** x_0

Step-wise construction of
Power Series Solutions of an ODE
Example

- Find power series solution of the ODE $y''+xy=0$
    - No singular points, hence power series solution with $x_o = 0$ guaranteed.

- **Step1**: Let $y = \sum_{n=0}^{\infty} c_n x^n$ be solution of the ODE.

- **Step2** $y' = \sum_{n=0}^{\infty} c_n n x^{n-1}$, and $y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$

- **Step2** The ODE becomes:

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)\, x^k + \sum_{k=1}^{\infty} c_{k-1}\, x^k$$

$$= 2c_2 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)\, x^k + \sum_{k=1}^{\infty} c_{k-1}\, x^k$$

$$= \sum_{k=2}^{\infty} \left[(k+2)(k+1)c_{k+2} + c_{k-1}\right] x^x + c_2$$

$$= 0$$

- **Step3** Compare the coefficients of ***constant*** term an the powers of x^k
    - $c_2 = 0$
    - $(k+2)(k+1)c_{k+2} + c_{k-1} = 0$

      $\Rightarrow c_{k+2} = -\dfrac{c_{k-1}}{(k+1)(k+2)} \forall k$

This relation is called a **Recursive relation**. It determines values of constants appearing in solution of the ODE. We see this in next steps.

• **Step4.** Use of $c_{k+2} = -\frac{c_{k-1}}{(k+1)(k+2)}$ to determine solution for different "**k**".

k	c_{k+2}	
1	$c_3 = -\frac{c_o}{2.3}$	
2	$c_4 = -\frac{c_1}{3.4}$	
3	$c_5 = -\frac{c_2}{4.5}$	$c_5 = 0 \because c_2 = 0$
4	$c_6 = -\frac{c_3}{5.6}$	$c_6 = \frac{1}{2.3.5.6} c_o$
5	$c_7 = -\frac{c_4}{6.7}$	$c_7 = \frac{1}{3.4.6.7} c_1$
6	$c_8 = -\frac{c_5}{7.8}$	$c_8 = 0 \because c_5 = 0$
7	$c_9 = -\frac{c_6}{8.9}$	$c_9 = \frac{1}{2.3.5.6.8.9} c_o$

• **Step5** Solution of the ODE becomes:

$$y = c_o + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + \cdots$$

$$y = c_o + c_1 x - \frac{c_o}{2.3} x^3 - c_1 \frac{c_1}{3.4} x^4 + \frac{c_o}{2.3.5.6} x^6 + \frac{c_1}{3.4.6.6} x^7 + \frac{c_0}{2.3.4.5.6.8.9} x^9 + \cdots$$

$$y = c_1 \left[x - \frac{1}{3.4} x^4 + \frac{1}{3.4.6.6} x^7 - \cdots \right] + c_o \left[1 - \frac{1}{2.3} x^3 + \frac{1}{2.3.5.6} x^6 - \frac{1}{2.3.4.5.6.8.9} x^9 + \cdots \right]$$

$$y = c_1 y_1 + c_o y_2$$

Exercise 6.1.2

Q 19 Find series solution of $y''-2xy'+y=0$ about the ordinary point $x=0$.

Solution:

- **Step1**: Let $y=\sum_{n=0}^{\infty} c_n x^n$ be solution of the ODE.

- **Step2** $y'=\sum_{n=1}^{\infty} c_n n x^{n-1}$, and $y''=\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$

- **Step2** The ODE becomes:

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - 2\sum_{n=1}^{\infty} c_n n x^n + \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)\,x^k - 2\sum_{k=1}^{\infty} c_k\,k\,x^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= \underbrace{2c_2 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)\,x^k}\; - 2\sum_{k=1}^{\infty} c_k\,k\,x^k + \underbrace{c_o + \sum_{k=1}^{\infty} c_k x^k}$$

$$= \sum_{k=1}^{\infty}\left[(k+2)(k+1)c_{k+2} - c_k(2k-1)\right]x^x + 2c_2 + c_o = 0$$

- **Step3** Compare coefficients to get:
  - $c_2 = -c_o/2$
  - $c_{k+2} = \dfrac{(2k-1)c_k}{(k+1)(k+2)}\ \forall k = 1,2,3\cdots$

- **Step4** Evaluation of coefficients from recursive relation

$$c_2 = -c_o/2$$

$$c_3 = \frac{c_1}{(2)(3)}$$

$$c_4 = \frac{3c_2}{(3)(4)} = -\frac{c_o}{2.4} = -\frac{c_o}{8}$$

$$c_5 = \frac{(5)c_3}{(4)(5)} = \frac{c_3}{4} = \frac{c_1}{2.3.4}$$

$$c_6 = \frac{(7)c_4}{(5)(6)} = -\frac{7}{5.6.8}c_o \text{ etc}$$

- **Step5** Plug the above values in the polynomial and writing solutions:

$$y = c_o + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \cdots$$

$$y = c_o + c_1 x - \frac{c_o}{2}x^2 + \frac{c_1}{6}x^3 - \frac{c_o}{8}x^4 + \frac{c_1}{24}x^5 - \frac{7c_o}{240}x^6 + \cdots$$

$$y = c_o(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 + \cdots) + c_1(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \cdots)$$

$$y = c_o y_0 + c_1 y_1$$

where

$$y_0 = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 + \cdots = 1 - \frac{1}{2!}x^2 - \frac{3}{4!}x^4 + \frac{3.7}{6!}x^6 + \cdots$$

and

$$y_1 = x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \cdots = x + \frac{3}{3!}x^3 + \frac{5}{5!}x^5 + \cdots$$

- **Step6** **Particular Solutions**

When $c_1 = 0$ we get one particular solution y_0

When $c_0 = 0$ we get one particular solution y_1

Q 27 Find series solution of $(x^2+2)y''+3xy'-y=0$ about the ordinary point $x=0$.

Solution:

- **Step1**: Let $y=\sum_{n=0}^{\infty} c_n x^n$ be solution of the ODE.
- **Step2** $y'=\sum_{n=1}^{\infty} c_n n x^{n-1}$, and $y''=\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$
- **Step2** The ODE becomes:

$$\sum_{n=2}^{\infty} c_n n(n-1)x^n + 2\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + 3\sum_{n=1}^{\infty} c_n n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

- **Step3** Change "$n-2$" to "k" in the middle term so that we can take "x^n" common

$$\sum_{k=2}^{\infty} c_k k(k-1)x^k + 2\underbrace{\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k}_{} + 3\underbrace{\sum_{k=1}^{\infty} c_k k\, x^k}_{} - \underbrace{\sum_{k=0}^{\infty} c_k x^k}_{} = 0$$

- **Step4** Open summation over "$k=0$" above as below

$$\sum_{k=2}^{\infty} c_k k(k-1)x^k + 4c_2 + 12c_3 x + 2\sum_{k=2}^{\infty} c_{k+2}(k+2)(k+1)x^k +$$

$$+3c_1 x + 3\sum_{k=2}^{\infty} c_k k\, x^k - c_0 - c_1 x - \sum_{k=2}^{\infty} c_k x^k = 0$$

- **Step5** Simplify above expression:

$$\sum_{k=2}^{\infty}[c_k k(k-1) + 2c_{k+2}(k+1)(k+2) + 3c_k k - c_k]x^k +$$
$$4c_2 + 12c_3 x + 2c_1 x - c_0 = 0$$

- **Step6** Compare coefficients of "x^k", "x" and constant term to get:

$$c_k(k^2+2k-1)+2c_{k+2}(k+1)(k+2)=0$$

$$\Rightarrow c_{k+2}=-\frac{k^2+2k-1}{2(k+1)(k+2)}c_k,\ k=2,3,\cdots$$

$$4c_2-c_0=0$$

$$6c_3+c_1=0$$

- **Step7** From the last two equations we get:

$$c_0=4c_2 \Rightarrow c_2=c_o/4$$

$$c_1=-6c_3 \Rightarrow c_3=-c_1/6$$

- **Step8** Use recursive relation x to determine solutions as follows:

  - $k=2 \Rightarrow c_4=-\frac{7}{24}c_2=\frac{-7}{96}c_0=\frac{-7}{4.4!}$
  - $k=3 \Rightarrow c_5=-\frac{7}{20}c_3=\frac{7}{120}c_1=\frac{7}{5!}c_1$
  - $k=4 \Rightarrow c_6=-\frac{23}{60}c_4=\frac{-23}{240}c_0=\frac{-23}{2.5!}c_0$

- **Step9** Writing solution

$$y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+\cdots$$

$$y=c_0+c_1x+\frac{c_o}{4}x^2-\frac{c_1}{6}x^3-\frac{7c_0}{96}x^4+\frac{7c_1}{120}x^5+\frac{-23c_0}{240}x^6+\cdots$$

$$y=c_0(1+\frac{1}{4}x^2-\frac{7}{96}x^4-\frac{23}{240}x^6+\cdots)+c_1(x-\frac{1}{6}x^3+\frac{7}{120}x^5-\cdots)$$

$$y=c_0y_1+c_1y_1$$

- **Step10** **Particular Solutions**

  When $c_1=0$ we get one particular solution y_0

  When $c_0=0$ we get one particular solution y_1

Q 31 Use power series method to solve $y''-2xy'+8y=0$ when $y(0)=0, y'(0)=0$

Solution:

- **Step1**: Let $y=\sum_{n=0}^{\infty} c_n x^n$ be series solution of the given ODE.

- **Step2** $y'=\sum_{n=1}^{\infty} c_n n x^{n-1}$, and $y''=\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$

- **Step2** The ODE becomes:

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - 2\sum_{n=1}^{\infty} c_n n x^n + 8\sum_{n=0}^{\infty} c_n x^n = 0$$

- **Step3** Do the following in above expression:
  - In first term: $n-2=k \Rightarrow n=k+2$
  - In 2nd term: $n=k$
  - In 3rd term: $n=k$

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - 2\sum_{k=1}^{\infty} c_k k x^k + 8\sum_{k=0}^{\infty} c_k x^k = 0$$

- **Step4** Open summation in **first** and **third** term up to k=0 as below:

$$2c_2 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k - 2\sum_{k=1}^{\infty} c_k k x^k + 8c_0 + 8\sum_{k=1}^{\infty} c_k x^k = 0$$

- **Step5** Re-write above as:

$$2c_2 + 8c_0 + \sum_{k=1}^{\infty}\left(c_{k+2}(k+2)(k+1) - 2c_k k + 8c_k\right)x^k = 0$$

- **Step6** Above gives:

$$c_{k+2} = \frac{2(k-4)}{(k+1)(k+2)}c_k \quad \forall k=1,2,3\cdots \text{ and}$$

$$c_2 = -4c_0$$

- **Step7** From recursive relation write coefficients as:
  - $k = 1 \Rightarrow c_3 = \frac{-6}{6} c_1 = -c_1$
  - $k = 2 \Rightarrow c_4 = \frac{-4}{12} c_2 = -\frac{1}{3}(-4c_0) = \frac{4}{3} c_0$
  - $k = 3 \Rightarrow c_5 = \frac{-2}{20} c_3 = -\frac{1}{10}(-c_1) = \frac{1}{10} c_1$
  - $k = 4 \Rightarrow c_6 = 0$

Since all of $c_8 = 0$, c_{10} etc will be expressed in terms of each other, they are all zero. We therefore evaluate only the odd terms only.

  - $k = 5 \Rightarrow c_7 = \frac{2}{42} c_5 = \frac{1}{21}(\frac{1}{10} c_1) = \frac{1}{210} c_1$ etc
- **Step8** Writing solution
  - $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$
  - $y = c_0 + c_1 x - 4c_0 x^2 - c_1 x^3 + \frac{4}{3} c_0 x^4 + \frac{1}{10} c_1 x^5 + \frac{1}{210} c_1 x^7 + \cdots$
  - $y = c_0 (1 - 4x^2 + \frac{4}{3} x^4) + c_1 (x - x^3 + \frac{1}{10} x^5 + \frac{1}{210} x^7 + \cdots)$
- **Step9** Use initial conditions
  - $y(0) = 3 \Rightarrow c_0 = 3$
    $y'(0) = 0 \Rightarrow c_1 = 0$
- **Step10** The solution of the IVP is:
  - $y = 3(1 - 4x^2 + \frac{4}{3} x^4)$

Q 34 Use power series method to solve $y''+e^{x}y'-y=0$ about ordinary point $x_0=0$

Solution:

- **Step1**: Let $y=\sum_{n=0}^{\infty} c_n x^n$ be series solution of the given ODE.

- **Step2** $y'=\sum_{n=1}^{\infty} c_n n x^{n-1}$, and $y''=\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$

- **Step3** Write series expression of e^{x} about $x_0=0$

  $$e^{x}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

- **Step4** Substitute above expressions in the ODE to write it as:

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}+(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots)\sum_{n=1}^{\infty} c_n n x^{n-1}-\sum_{n=0}^{\infty} c_n x^n=0$$

- **Step5** Expand the above expression as:

$$(2c_2+6c_3x+12c_4x^2+20c_5x^3+\cdots)+(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots)(c_1+2c_2x+3c_3x^2+\cdots)$$
$$-(c_0+c_1x+c_2x^2+c_3x^3+\cdots)=0$$

- **Step6** Re-write above as:

$$(2c_2+c_1-c_0)+(6c_3+2c_2+c_1-c_1)x+(12c_4+3c_3+2c_2+\frac{c_1}{2})x^2+\cdots=0$$ •

Step7 Compare and get:

$$2c_2 + c_1 - c_0 = 0$$
$$\Rightarrow c_2 = (c_0 - c_1)/2$$

$$3c_3 + c_2 = 0$$
$$\Rightarrow c_3 = -c_2/2 = (c_0 - c_1)/4$$

$$12c_4 + 3c_3 + 2c_2 + \frac{c_1}{2} = 0$$
$$\Rightarrow c_4 = -(c_{1/2} + 2c_2 + 3c_3)/12$$
$$\Rightarrow c_4 = -\left[\frac{c_1}{2} + c_0 - c_1 - \frac{c_0}{2} + \frac{c_1}{2}\right]/12$$
$$\Rightarrow c_4 = -\left[\frac{c_0}{2} - c_1\right] = c_1 - c_0/2$$

Step8 Substitute above values in

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

and re-arrange terms to get the answer.