6.1.1 Review of Power Series

Objectives:

- Review of Power Series
- Review of Convergence, radius of convergence, absolute convergence of such series
- Review of Ratio Test to test the convergence of power series
- Shifting the index of summation
- Analytical points
- Taylor and Maclaurin series

An infinite series in $x - x_0$ of the form

$$f(x) = \sum_{n=0}^{\infty} c_n \left(x - x_0\right)^n = c_0 + c_1\left(x - x_0\right) + c_2\left(x - x_0\right)^2 + \cdots\cdots + c_n\left(x - x_0\right)^n + \cdots$$

is called a ***power series***. The number x_0 is called the ***centre*** of the series.

Convergence of a Power Series: A power series

$$\sum_{n=0}^{\infty} c_n \left(x - x_0\right)^n = c_0 + c_1\left(x - x_0\right) + c_2\left(x - x_0\right)^2 + \cdots\cdots + c_n\left(x - x_0\right)^n + \cdots$$

is said to be ***convergent*** at a point x if its sequence of partial sums $\{S_N(x)\}$ convergences, that is, if

$$\lim_{N \to \infty} S_N(x) = \lim_{N \to \infty} \sum_{n=0}^{\infty} c_n (x - x_0)^n \quad \text{exists.}$$

An infinite series in $x - x_0$ of the form

$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ is said to be ***converges absolutely*** if

$|f(x)| = \left| \sum_{n=0}^{\infty} c_n (x - x_0)^n \right|$ converges.

Convergence of Power series: Given a power series in $x - x_0$. Then either

1) it converges only for $x = x_0$.

OR

2) it converges for all values of x

OR

3) it converges for x in an open interval $(x_0 - R, x_0 + R)$ and diverges if $x < x_0 - R$, $x > x_0 + R$

- At the end points $x = x_0 - R$ or $x = x_0 + R$, the series may converge or diverge. (we need to investigate separately at the end points)

Interval of convergence
||
all values for which series converges

Radius of convergence
||
half of length of interval of convergence

Ratio Test for checking Convergence of a Power Series:

- **Convergence of a power series**

  $f(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^n$ can be tested by ratio test as follows:

- **Construct the nth term:** $a_n = c_n (x-x_0)^n$

- **Construct the (n+1)th term as:** $a_{n+1} = c_{n+1}(x-x_0)^{n+1}$

- **Construct the ratio:** $\dfrac{a_{n+1}}{a_n} = \dfrac{c_{n+1}(x-x_0)^{n+1}}{c_n (x-x_0)^n}$

- **Construct the absolute value:**

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0| \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$$

- **If** $L<1$, then the series converges absolutely hence converges

  If $L>1$, then the series diverges

  If $L=1$, then the series may converge or diverge.

Question 3/248: Use ratio test, find the radius of convergence and interval of convergence of the series $f(x)=\sum_{k=1}^{\infty}\frac{(-1)^k}{10^k}(x-2)^k$

Differentiation and Integration of power series (term by term):

Let $f(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^n$ with radius of convergence R.

Then in the radius of convergence

$$f'(x)=\sum_{n=0}^{\infty} \frac{d}{dx}\left(c_n (x-x_0)^n\right)$$

and

$$\int f(x)dx=\sum_{n=0}^{\infty} \int \left(c_n (x-x_0)^n\right)dx.$$

Both have radius of convergence R.

Another fact

- Can add, subtract, multiply power series to get new power series.
- The sum, difference, product have same radius of convergence.

Shifting (or changing) index of a summation

Important tool

We will need to make changes of summation indices in calculating series solutions of differential equations

- Nothing new.
- An idea similar to change of variables in integration

Analytical Functions:

- Not every function can be expressed as power series.
- Those which can be expressed as power series are given a special name.

A function $f(x)$ is said to be analytic at a point x_0 if it can be represented as a power series with a positive radius of convergence.

i.e. $f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$

Examples "Analytic"

- $f(x) = e^x$
- $f(x) = \sin x$
- $f(x) = ax^2 + bx + c$

Example "Non-Analytic"

- $f(x) = \dfrac{1}{x-1}$ at $x = 1$

- All polynomials are analytic
- If $P(x), Q(x)$ are polynomials with no common factors.

Then $\dfrac{P(x)}{Q(x)}$ is analytic at points where $Q(x) \neq 0$

Taylor and Maclaurin Series:

Gives a way of finding power series of analytic functions

If $f(x)$ is analytic at x_0 then the Taylor's Series of a function $f(x)$ about x_0 is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$
$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

Taylor series evaluated at $x_0 = 0$ gives ***Maclaurin series***

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Important Basic Series:

Series	Interval of convergence
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots\cdots$	$-1 < x < 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$	$-\infty < x < \infty$
$\sin x = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$	$-\infty < x < \infty$
$\cos x = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots$	$-\infty < x < \infty$
$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\cdots$	$-1 \le x \le 1$

Question 5/248: Find first four terms of the series $f(x)=\sin x\cos x$ at $x_0=0$.

Answer:

- $f(x)=\sin x\cos x$ $\qquad f(x)|_{x=0}=0$
- $f'(x)=\cos 2x$ $\qquad f'(x)|_{x=0}=1$
- $f''(x)=-2\sin 2x$ $\qquad f''(x)|_{x=0}=0$
- $f'''(x)=-4\cos 2x$ $\qquad f'''(x)|_{x=0}=-4$
- $f^{(4)}(x)=8\sin 2x$ $\qquad f^{(4)}(x)|_{x=0}=0$
- $f^{(5)}(x)=16\cos 2x$ $\qquad f^{(5)}(x)|_{x=0}=16$

$$f(x)=\frac{0}{0!}+\frac{x}{1!}+\frac{0x^2}{2!}+\frac{-4x^3}{3!}+\frac{0x^4}{4!}+\frac{16x^5}{5!}+\cdots$$

$$=x-\frac{2}{3}x^3+\frac{2x^5}{15}+\cdots$$

Question 9/248: Rewrite the series $\sum_{n=1}^{\infty} nc_n x^{n+2}$ so that the general terms involves x^k.

Answer: Define $n+2=k$

Write $n=k-2$ in the series to get

$$\sum_{k=3}^{\infty}(k-2)c_{k-2}x^k$$

Question 12/248: Rewrite the expression

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3\sum_{n=1}^{\infty} nc_n x^n$$

as a single series whose general terms involves x^k.

Answer:

Question 13/248: Verify that $y=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ is a particular solution of ODE $(x+1)y''+y'=0$.

Answer:

- Differentiate twice y

$$y'=\sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}$$

$$y''=\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2}$$

- Substitute above values in the left hand side of the ODE and simplify

$$(x+1)y''+y'=\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-1}+\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2}$$

$$+\sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}$$

$$=\underbrace{\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-1}}_{n-1=k\quad\therefore n=k+1}+\underbrace{\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2}}_{n-2=k\quad\therefore n=k+2}$$

$$+\underbrace{\sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}}_{n-1=k\quad\therefore n=k+1}$$

$$=\sum_{k=1}^{\infty}(-1)^{k+2}(k)x^k+\sum_{k=0}^{\infty}(-1)^{k+3}(k+1)x^k+\sum_{k=0}^{\infty}(-1)^{k+2}x^k$$

$$=\sum_{k=1}^{\infty}(-1)^{k+2}(k)x^k-1+\sum_{k=1}^{\infty}(-1)^{k+3}(k+1)x^k+1+\sum_{k=1}^{\infty}(-1)^{k+2}x^k$$

$$=\sum_{k=1}^{\infty}(-1)^{k+2}[k-(k+1)+1]x^k=0$$