4.7 Cauchy-Euler Equation

Cauchy-Euler Equation: An ODE of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x), \qquad (1)$$

where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants, is called

Cauchy-Euler ODE .

- Since the coefficient of $y^{(n)}$ is zero at $x = 0$, we confine our attention to finding the general solutions defined on the interval $(0, \infty)$. Solutions on the interval $(-\infty, 0)$ can be obtained by substituting $t = -x$ into the ODE.
- The present method works for equations of all order, but here we only focus on second order ODE.

Method of Finding Solution of $ax^2y'' + bxy' + cy = 0$ (2)

- **Step 1**. Let $y = x^m$ be a solution of ODE (2).
- **Step 2**. Then $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$
- **Step 3**. Substitute above values in ODE (2)

$$ax^2 m(m-1)x^{m-2} + bxmx^{m-1} + cx^m = 0$$

or $$[am(m-1) + bm + c]x^m = 0$$

or $$am^2 + (b-a)m + c = 0 \qquad (3)$$

- **Step 4**. Roots of Equation (3)

Case 1. Real and Distinct (m_1, m_2) — Discriminant is positive

Case 2. Real and Repeated (m_1, m_1) — Discriminant is zero

Case 3. Complex Roots $\alpha \pm i\beta$ — Discriminant is negative

CASE 1

Solution of the Cauchy-Euler Equation (Real distinct roots)

$$y(x) = c_1 x^{m_1} + c_2 x^{m_2}$$

Question 1/178: Solve $x^2y'' - 6y = 0$.

CASE 2

Solution of the Cauchy-Euler Equation (Real & repeated roots)

- Re-write the ODE as : $y'' + \frac{b}{ax} y' + \frac{c}{ax^2} y = 0$
- One solution of (3) is $y_1 = x^{m_1}$, where $m_1 = (a-b)/2a$
- Second solution of (3) is

$$y_2 = y_1 \int \frac{e^{-\int \frac{b}{ax} dx}}{y_1^2} dx$$

Consult method of finding second solution given one

$$= x^{m_1} \int \frac{x^{-b/a}}{x^{2m_1}} dx = x^{m_1} \int \frac{x^{-b/a}}{x^{\left(\frac{a-b}{a}\right)}} dx$$

$$= x^{m_1} \int \frac{dx}{x} = x^{m_1} \ln x$$

- General solution of ODE is:

$$y = x^{m_1}(c_1 + c_2 \ln x).$$

Question 12/178: Solve $x^2 y'' + 8xy' + 6y = 0$.

CASE 3

Solution of the Cauchy-Euler Equation (Complex Conjugate roots)

- Re-write the ODE as : $y'' + \frac{b}{ax} y' + \frac{c}{ax^2} y = 0$
- Solution of (3) is $m_1 = \alpha + i\beta$, where $m_2 = \alpha - i\beta$
- The general solution:

$$
\begin{aligned}
y &= c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta} \\
&= c_1 x^{\alpha} x^{i\beta} + c_2 x^{\alpha} x^{-i\beta} \\
&= c_1 x^{\alpha} \left(e^{\ln x}\right)^{i\beta} + c_2 x^{\alpha} \left(e^{\ln x}\right)^{-i\beta} \\
&= c_1 x^{\alpha} \left(e^{i\beta \ln x}\right) + c_2 x^{\alpha} \left(e^{-i\beta \ln x}\right) \\
&= \left[c \cos(\beta \ln x) + d \sin(\beta \ln x)\right]
\end{aligned}
$$

Final form of general solution when roots of auxiliary Equ. are complex

Question 13/178: Solve $3x^2 y'' + 6xy' + y = 0$.

Cauchy-Euler Equation $ax^2y'' + bxy' + cy = 0$

Reducing to Constant Coefficient Method

- Substitute $x = e^t \Rightarrow \dfrac{dx}{dt} = e^t$
- Evaluate $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \left(\frac{dy}{dt}\right)\frac{1}{x}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \left(\frac{dy}{dt}\right)\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x}\left(\frac{d^2y}{dt^2}\frac{1}{x}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

- Substitute above results in ODE

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t$$

This last equation is called a constant coefficients equation.

Question 32/178: Solve $x^2y'' - 9xy' + 25y = 0$ by reducing it to constant coefficient equation.

Question 21/178: Solve $x^2y'' - xy' + y = 2x$ by variation of parameter method.

Question 25/178: Solve the IVP $x^2y'' + 4xy' = 0$ subject to $y(1) = 0,\ \ y'(1) = 6$.