4.6 Variation of Parameters

Objectives:

- Not always possible to annihilate RHS of the non-homogeneous ODE using annihilator approach (e.g. $\ln x$, $\sec x$)
- Such cases of non-homogeneous ODEs can be solved using variation of parameter method.
- A method for finding particular solution y_p of non-homogeneous ODE, if we know the complementary solution y_c of associated homogeneous equation.

Key idea of the method

- Vary the constants c_1, c_2 in y_c

  that is, replace c_1, c_2 **with functions** $u_1(x), u_2(x)$ **in** y_c
- and try to find y_p of the form $y_p = u_1(x)y_1 + u_2(x)y_2$

Next we look at main question:

How to find $u_1(x), u_2(x)$

How to find $u_1(x),\ u_2(x)$

* If we know the complimentary solution

$$y_c = c_1 y_1 + c_2 y_2$$

of a homogeneous equation $y'' + p(x)y' + q(x)y = 0$

* then a particular solution y_p of the non-homogeneous equation

$$y'' + p(x)y' + q(x)y = f(x)$$

is given by

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

* where the functions $u_1(x),\ u_2(x)$ can be found by integrating

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$

Can you see why W is never zero?

- and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Wronskian W of y_1, y_2

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}$$

1st column of W replaced by $\begin{matrix} 0 \\ f \end{matrix}$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}$$

2nd column of W replaced by $\begin{matrix} 0 \\ f \end{matrix}$

Procedure for solving by "Variation of Parameters"

- **Question:** To find general solution of a 2nd order non-homogeneous linear differential equation

Importan
Leading coefficient must be 1

* Write equation in the form

$$y'' + p(x)y' + q(x)y = f(x) \quad \textbf{(*)}$$

* Find complimentary solution $y_c = c_1 y_1 + c_2 y_2$ by solving

$$y'' + p(x)y' + q(x)y = 0$$

* Set $y_p = u_1(x) y_1 + u_2(x) y_2$ **(1)**

* Find $u_1(x),\ u_2(x)$ by integrating $u_1' = \dfrac{W_1}{W}, \quad u_2' = \dfrac{W_2}{W}$

where

Wronskian W of y_1, y_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

1st column of W replaced by $\begin{matrix} 0 \\ f \end{matrix}$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix} = -y_2 f$$

2nd column of W replaced by $\begin{matrix} 0 \\ f \end{matrix}$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix} = y_1 f$$

In other words,

$$u_1' = \frac{W_1}{W} = \frac{y_2 f}{y_1 y_2' - y_1' y_2} \Rightarrow u_1 = \int \frac{y_2 f}{y_1 y_2' - y_1' y_2} dx$$

and

$$u_2' = \frac{W_2}{W} = \frac{y_1 f}{y_1 y_2' - y_1' y_2} \Rightarrow u_2 = \int \frac{y_1 f}{y_1 y_2' - y_1' y_2} dx$$

* Write y_p using Equation (1)

Write general solution of (*) as $y = y_c + y_p$

Question 12/172: Solve $y'' + 3y' + 2y = e^x/(1+e^x)$.

Question 17/172: Solve $3y'' - 6y' + 6y = e^x \sec x$.

Question 19/172: Solve the IVP $4y'' - y = xe^{x/2}$ subject to $y(0) = 1,\ y'(0) = 0$.

Question 23/172: Find the general solution of the non-homogeneous ODE $x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = x^{3/2}$.