4.5 Undetermined Coefficients – Annihilator Approach

Objectives:

- To learn a method of solving non homogeneous ODEs with constants coefficients using annihilator approach.
- This approach converts right hand side of the ODE to zero (what means by annihilation).
- We then find roots of the resulting ODE by the methods adopted in Section 4.3.

Recall the non-homogeneous linear differential equation with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) \qquad a_n \neq 0 \quad (*)$$

where $f(x)$ is a

- polynomial
- or exponential function
- or sine function
- or cosine function
- or a product (&/or sum) of some (or all) of these

Key idea of the method

Look at the form of $f(x)$. By using a linear differentiable operator, we make the left hand side of (*) zero.

Annihilator Operator: A linear differentiable operator L with constants is called annihilator if $L(f(x)) = 0$.

Operators annihilating polynomial functions

Function	Construction of Operator Annihilating the function	Annihilator
α	$D(\alpha) = 0$	D
x	$D(x) = 1 \Rightarrow D(D(x)) = D(1) = 0$	D^2
$1+x$	$D(1+x) = 1 \Rightarrow D(D(x+1)) = D(1) = 0$ $\Rightarrow D^2(1+x) = 0$	D^2
$1+x+x^2$	$D(1+x+x^2) = 1+2x \Rightarrow D(D(1+x+x^2)) = D(1+2x) = 2$ $\Rightarrow D(D(D(1+x+x^2))) = D(2) = 0 \Rightarrow D^3(1+x+x^2) = 0$	D^3
$1+x+\cdots+x^n$	$D^{n+1}(1+x+\cdots+x^n) = 0$	D^{n+1}

A polynomial of degree n is annihilated by the derivative operator of order $n+1$ given by D^{n+1}

Operators annihilating exponential functions

Function	Construction of Operator Annihilating the function	Annihilator
e^{x}	$De^{x}=e^{x}\Rightarrow De^{x}-e^{x}=0\Rightarrow(D-1)e^{x}=0$	$D-1$
e^{2x}	$De^{2x}=2e^{2x}\Rightarrow De^{2x}-2e^{2x}=0\Rightarrow(D-2)e^{2x}=0$	$D-2$
2^{3x}	$De^{3x}=3e^{3x}\Rightarrow De^{3x}-3e^{3x}=0\Rightarrow(D-3)e^{3x}=0$	$D-3$
e^{nx}	$De^{nx}=ne^{nx}\Rightarrow De^{nx}-ne^{nx}=0\Rightarrow(D-n)e^{nx}=0$	$D-n$

An exponential e^{nx} is annihilated by the derivative operator $D-n$

Operators annihilating $\cos nx\,/\sin nx$ functions

Function	Construction of Operator Annihilating the function	Annihilator
$\cos x$	$D\cos x=-\sin x\Rightarrow D(D\cos x)=-\cos x$ $\Rightarrow D^{2}\cos x+\cos x=0\Rightarrow(D^{2}+1)\cos x=0$	$D^{2}+1$
$\cos 2x$	$D\cos 2x=-2\sin x\Rightarrow D(D\cos 2x)=-4\cos 2x$ $\Rightarrow D^{2}\cos 2x+4\cos 2x=0\Rightarrow(D^{2}+4)\cos x=0$	$D^{2}+4$
$\cos nx$	$D\cos nx=-n\sin x\Rightarrow D(D\cos nx)=-n^{2}\cos nx$ $\Rightarrow D^{2}\cos nx+n^{2}\cos nx=0\Rightarrow(D^{2}+n^{2})\cos nx=0$	$D^{2}+n^{2}$

$\cos nx\,/\sin nx$ is annihilated by the derivative operator $D^{2}+n^{2}$

Operators annihilating mixed exponential & polynomial functions

Function	Construction of Operator Annihilating the function	Annihilator
$xe^{\alpha x}$	$D(xe^{\alpha x}) = e^{\alpha x} + \alpha xe^{\alpha x} \Rightarrow D(xe^{\alpha x}) - \alpha xe^{\alpha x} = e^{\alpha x}$ $(D-\alpha)xe^{\alpha x} = e^{\alpha x}$ $Note \quad (D-\alpha)e^{\alpha x} = 0$ $\therefore (D-\alpha)^2 xe^{\alpha x} = 0$	$(D-\alpha)^2$
$x^2e^{\alpha x}$	$D(x^2e^{\alpha x}) = 2xe^{\alpha x} + \alpha x^2e^{\alpha x} \Rightarrow D(x^2e^{\alpha x}) - \alpha x^2e^{\alpha x} = 2xe^{\alpha x}$ $(D-\alpha)x^2e^{\alpha x} = 2xe^{\alpha x}$ $Note \quad (D-\alpha)^2 xe^{\alpha x} = 0$ $\therefore (D-\alpha)^3 x^2e^{\alpha x} = 0$	$(D-\alpha)^3$
$x^3e^{\alpha x}$	$D(x^3e^{\alpha x}) = 3xe^{\alpha x} + \alpha x^3e^{\alpha x} \Rightarrow D(x^3e^{\alpha x}) - \alpha x^3e^{\alpha x} = 3x^2e^{\alpha x}$ $(D-\alpha)x^3e^{\alpha x} = 3x^2e^{\alpha x}$ $Note \quad (D-\alpha)^3 x^2e^{\alpha x} = 0$ $\therefore (D-\alpha)^4 x^3e^{\alpha x} = 0$	$(D-\alpha)^4$
$x^ne^{\alpha x}$	$D(x^ne^{\alpha x}) = nx^{n-1}e^{\alpha x} + \alpha x^ne^{\alpha x} \Rightarrow D(x^ne^{\alpha x}) - \alpha x^ne^{\alpha x} = nx^{n-1}e^{\alpha x}$ $(D-\alpha)x^ne^{\alpha x} = nx^{n-1}e^{\alpha x}$ $Note \quad (D-\alpha)^n x^{n-1}e^{\alpha x} = 0$ $\therefore (D-\alpha)^{n+1} x^ne^{\alpha x} = 0$	$(D-\alpha)^{n+1}$

A mixed exponential and polynomial $x^ne^{\alpha x}$ is annihilated by the derivative operator $(D-\alpha)^{n+1}$

Operators annihilating mixed exponential, polynomial & trigonometric functions

Function	Construction of Operator Annihilating the function	Annihilator
$e^{\alpha x}\cos\beta x$	$D(e^{\alpha x}\cos\beta x) = \alpha e^{\alpha x}\cos\beta x - \beta e^{\alpha x}\sin\beta x$ $D^2(e^{\alpha x}\cos\beta x) = \alpha^2 e^{\alpha x}\cos\beta x - 2\alpha\beta e^{\alpha x}\sin\beta x$ $-\beta^2 e^{\alpha x}\cos\beta x$ $(D^2 - 2\alpha D)(e^{\alpha x}\cos\beta x) = -2\alpha\beta e^{\alpha x}\sin\beta x$ $-\beta^2 e^{\alpha x}\cos\beta x$ $(D^2 - 2\alpha D + \alpha^2 + \beta^2)(e^{\alpha x}\cos\beta x) = 0$	$D^2 - 2\alpha D$ $+\alpha^2 + \beta^2$
$xe^{\alpha x}\cos\beta x$	$(D^2 - 2\alpha D + \alpha^2 + \beta^2)^2(xe^{\alpha x}\cos\beta x) = 0$	$[D^2 - 2\alpha D$ $+\alpha^2 + \beta^2]^2$
$x^2 e^{\alpha x}\cos\beta x$	$(D^2 - 2\alpha D + \alpha^2 + \beta^2)^3(xe^{\alpha x}\cos\beta x) = 0$	$[D^2 - 2\alpha D$ $+\alpha^2 + \beta^2]^3$
$x^n e^{\alpha x}\cos\beta x$	$(D^2 - 2\alpha D + \alpha^2 + \beta^2)^{n+1}(xe^{\alpha x}\cos\beta x) = 0$	$[D^2 - 2\alpha D$ $+\alpha^2 + \beta^2]^{n+1}$

A mixed exponential, polynomial and trigonometric function $x^n e^{\alpha x}\cos\beta x$ is annihilated by $[D^2 - 2\alpha D + \alpha^2 + \beta^2]^{n+1}$.

Similarly, a mixed exponential, polynomial and trigonometric function $x^n e^{\alpha x}\sin\beta x$ is annihilated by $[D^2 - 2\alpha D + \alpha^2 + \beta^2]^{n+1}$.

Summary of Annihilators

Polynomial of degree n annihilated by D^{n+1}

An exponential e^{nx} is annihilated by the derivative operator $D - n$

An exponential e^{nx} is annihilated by the derivative operator $D - n$

$\cos nx$ / $\sin nx$ is annihilated by the derivative operator $D^2 + n^2$

A mixed exponential and polynomial $x^n e^{\alpha x}$ is annihilated by the derivative operator $(D - \alpha)^{n+1}$

A mixed exponential, polynomial and trigonometric function $x^n e^{\alpha x} \cos \beta x$ is annihilated by $[D^2 - 2\alpha D + \alpha^2 + \beta^2]^{n+1}$.

A mixed exponential, polynomial and trigonometric function

$x^n e^{\alpha x} \sin \beta x$ is annihilated by $[D^2 - 2\alpha D + \alpha^2 + \beta^2]^{n+1}$.

Question 6/166: Write $y''' + 4y' = e^{x} \cos 2x$ in operator form the annihilated ODE.

Question 7/166: Write $y''' + 2y'' - 13y' + 10y = xe^{-x}$ in operator form the annihilated ODE.

Question 14/166: Verify that $D^{2} + 64$ annihilates $y = 2\cos 8x - 5\sin 8x$.

Question 25/167: Find a linear differential operator that annihilates $3 + e^{x} \cos 2x$.

Question 27/167: Find linearly independent functions that are annihilated by D^{5}.

Question 33/167: Find linearly independent functions that are annihilated by $D^{3} - 10D^{2} + 25D$.

Question 39/167: Solve $y'' + 4y' + 4y = 2x + 6$.

Question 56/167: Solve $y'' + y = 4\cos x - \sin x$.

Question 60/167: Solve the IVP $y'' - 2y' + 5y = e^{x} \sin x$ such that $y(0) = 1, \quad y'(0) = 2$.