4.3 Homogeneous Linear ODEs with Constant Coefficients

- Consider a second order homogeneous linear ODE

$$ay'' + by' + cy = 0 \qquad a \neq 0 \tag{1}$$

- Assume that $y = e^{mx}$ is a solution of (1). Then (1) gives

$$\left[am^2 + bm + c\right]e^{mx} = 0 \quad \Rightarrow \quad am^2 + bm + c = 0$$

- The ***auxiliary equation*** of (1) is given by

$$am^2 + bm + c = 0 \tag{2}$$

Main idea for solving $ay'' + by' + cy = 0; \;\; a \neq 0$

1 **Write the auxiliary equation corresponding to the given ODE**

$$am^2 + bm + c = 0 \tag{2}$$

2 **Find the roots of auxiliary equation.**

Since the auxiliary equation is a quadratic equation, its roots are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

and we have three possibilities:

- **2 distinct real roots** [when $b^2 - 4ac > 0$]
- **1 real (repeated) root** [when $b^2 - 4ac = 0$]
- **2 complex conjugate roots** [when $b^2 - 4ac < 0$]

Complex roots always occur in conjugate pairs.
Can you see why?

Case 1. Solution of ODE (1) when roots of (2) are distinct

- Two ***distinct roots*** are: m_1 and m_2
- Two ***distinct linearly independent solutions*** are:

  $y_1(x) = e^{m_1 x}$ and $y_2(x) = e^{m_2 x}$
- The ***general solution*** of (1) is a linear combination of these linearly independent solutions, that is,

$$y(x) = \alpha e^{m_1 x} + \beta e^{m_2 x}$$

Case 2. Solution of ODE (1) when (real) roots of (2) are same

- Two ***repeated roots*** are: m_1 and m_1
- Two ***linearly independent solutions*** are: $y_1(x) = e^{m_1 x}$ and

  $y_2(x) = x e^{m_1 x}$
- The ***general solution*** of (1) is a linear combination of these linearly independent solutions, that is,

  $y(x) = \alpha e^{m_1 x} + \beta x e^{m_1 x}$ or $y(x) = (c_1 + c_2 x) e^{m_1 x}$

Case 3. Solution of ODE (1) when roots of (2) are complex

- Two ***complex roots*** are: $m_1 = \dfrac{-b + i\sqrt{b^2 - 4ac}}{2a}$ and

  $m_2 = \dfrac{-b - i\sqrt{b^2 - 4ac}}{2a}$

- Re-write above complex ***roots*** as: $m_1 = \alpha + i\beta$ and

  $m_2 = \alpha - i\beta$

- Two ***linearly independent solutions*** are:

  $y_1(x) = \alpha e^{(\alpha + i\beta)x}$ and $y_2(x) = \alpha e^{(\alpha - i\beta)x}$

- The ***general solution*** of (1) is a linear combination of these linearly independent solutions, that is,

$$y(x) = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

Recall
$e^{i\theta} = \cos\theta + i\sin\theta$

$$= c_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + c_2 e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

$$= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

Question 5/147: Find the general solution of $y'' - 10y' + 25y = 0$.

Question 8/147: Find the general solution of $y'' + 4y' - y = 0$.

Question 14/147: Find the general solution of $2y'' - 3y' + 4y = 0$.

Question 32/148: Solve the initial value problem $4y'' - 4y' - 3y = 0$ subject to $y(0) = 1, \ \ y'(0) = 5$.

Question 38/148: Solve the boundary value problem $y'' + 4y = 0$ subject to $y(0) = 0, \ \ y(\pi) = 0$.

Solving higher order homogeneous linear equations with constant coefficients

- **General form of nth order homogeneous linear ODE:**

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0; \qquad a_n \neq 0 \qquad \textbf{(*)}$$

- **Similar to second order ODE we have** the method of finding solutions of Eq. **(*)** depends on finding roots of an algebraic equation defined below.

- **The auxiliary equation** of (*) is given by

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0 \qquad \textbf{(**)}$$

- Recall:
  - We only need to find *n* linearly independent solutions $y_1(x),\ y_2(x), \cdots y_n(x)$ of (*) and then the general solution of (*) is $y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$

Main idea for solving

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0; \qquad a_n \neq 0$$

1 **Write the auxiliary equation.**

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0 \qquad (**)$$

2 **Find the roots of auxiliary equation.**

There can be

- **distinct real roots**
- **repeated real roots**
- **distinct complex roots**
- **repeated complex roots**

Note: For $n > 2$, we can have different kinds of roots in one question.

e.g. The auxiliary equation

$(m-1)(m-3)^2(m^2+1)=0$

has a distinct real root, a repeated real root and distinct complex roots.

3 **For each root, write the corresponding term(s) of the general solution of (*)**

Distinct real roots

If (**) has n distinct roots $m_1, m_2, ..., m_n$, then the

solutions of (*) are: $y_1 = e^{m_1x}, y_2 = e^{m_2x}, ..., y_n = e^{m_nx}$

General solution of (*) is:

$$_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_3e^{m_nx}$$

Repeated real roots

If (**) has k distinct roots $m_1, m_2, ..., m_k$, and $l = n - k$

repeated roots then solutions of (*) corresponding to k distinct roots

re: $y_1 = e^{m_1x}, y_2 = e^{m_2x}, ..., y_k = e^{m_kx}$

Solutions of (*) corresponding to $n - k$ repeated roots are:

$$y_1 = xe^{m_1x}, y_2 = x^2e^{m_2x}, ..., y_{n-k} = x^{(n-k)}e^{m_{n-k}x}$$

General solution of (*) is:

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_3e^{m_nx} + (d_1 + d_2x + \cdots + d_lx^{l-1})e^{mx}$$

A root which repeats k times is called a root of multiplicity k .

Distinct complex roots

For each distinct (conjugate) pair $r = \alpha \pm i\beta$ of complex roots, there will be terms of the form $c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$ in the general solution.

Repeated complex roots

For each (conjugate) pair $r = \alpha \pm i\beta$ of complex roots with multiplicity k, there will be $2k$ terms of the form

$$e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x,\ x^2 e^{\alpha x}\cos\beta x,\ \cdots,\ x^{k-1}e^{\alpha x}\cos\beta x$$

$$e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x,\ x^2 e^{\alpha x}\sin\beta x,\ \cdots,\ x^{k-1}e^{\alpha x}\sin\beta x$$

in the general solution.

Question 17/147: Find the general solution of

$$y''' + 5y'' + 3y' - 9y = 0.$$

Question 23/147: Find the general solution of $y^{(4)} + y''' + y'' = 0$.

Question 36/148: Solve the initial value problem

$$y''' + 2y'' - 5y' - 6y = 0 \text{ subject to } y(0) = y'(0) = 0,\ \ y''(0) = 1.$$