4.1.3 Non-Homogeneous Equations

- Consider non-homogeneous ODE

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+.....+a_1(x)\frac{dy}{dx}+a_0(x)y=f(x), \quad a_n(x)\neq 0$$

- The operator form of non-homogeneous ODE is

$$Ly=f(x) \qquad (*)$$

- The associated homogeneous ODE is

- $$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+.....+a_1(x)\frac{dy}{dx}+a_0(x)y=0, \quad a_n(x)\neq 0$$

  The operator form of non-homogeneous ODE is

$$Ly=0 \qquad (**)$$

Determining general solution of non-homogeneous ODE requires:

- Find the general solution of associated homogeneous ODE (**):

$$y_c=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)$$

called complementary solution

- Find a particular solution of (*): y_p

Particular solution y_p does not involve any arbitrary constant

- Then the general solution of (*) is given by

$$y=y_c+y_p$$

General solution

Example: The complementary and particular solutions of the non-homogeneous ODE $y'' - 3y' + 2y = x$ are:

- $y_c = \alpha e^{2x} + \beta e^{x}$ and
- $y_p = -2 + x$

Note arbitrary constants appearing in the complementary solution

Note **NO** arbitrary constants appearing in the particular solution

Superposition Principle:

- Let $Ly = f_1(x) + f_2(x) + \cdots + f_n(x)$ be an nth order Non-homogeneous ODE.
- Corresponding to each term on the right hand side there exists one Particular solution given by $y_{p_1}(x), y_{p_2}(x), \cdots, y_{p_n}(x)$.
- Let the complementary solution of $Ly = 0$ be

  $y_c = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$
- Then the ***superposition principle*** says that:

  $$y = y_c + b_1 y_{p_1}(x) + b_2 y_{p_2}(x) + \cdots + b_n y_{p_n}(x)$$

  is a ***general solution*** of ODE.

Example: The complementary and particular solutions of the non-homogeneous ODE $y'' - 3y' + 2y = 1 + x + xe^{x} + e^{2x}$ are:

- $y_c = \alpha e^{2x} + \beta e^{x}$ and

  Note arbitrary constants appearing in the complementary solution

- $y_{p_1} = (5/4) + (1/2)x$ is a particular solution of

  $y'' - 3y' + 2y = 1 + x$

- $y_{p_2} = -x\left(1 + \frac{1}{2}x\right)e^{x}$ is a particular solution of

  $y'' - 3y' + 2y = xe^{x}$

- $y_{p_3} = (1/6)e^{2x}$ is a particular solution of $y'' - 3y' + 2y = e^{2x}$

The general solution (using superposition principle) is:

$y = \alpha e^{2x} + \beta e^{x} + b_1 y_{p_1} + b_2 y_{p_2} + b_3 y_{p_3}$

Question 31/138: Verify that $y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$ is a solution of $y'' - 7y' + 10y = 24e^{x}$ on the interval $(-\infty, \infty)$.

Question 35(a)/138: Verify that $y_p = x^2 + 3x$ is a particular solution of $y'' - 6y' + 5y = 5x^2 + 3x - 16$ on the interval $(-\infty, \infty)$.

Question 32(b)/138: Verify that $y_{p_1} = 3e^{2x}$ and $y_{p_2} = x^2 + 3x$ are solutions of $y'' - 6y' + 5y = -9e^{2x}$ and

$y'' - 6y' + 5y = 5x^2 + 3x - 16$, respectively, on the interval $(-\infty, \infty)$.