4.1.2 Homogeneous Equations

Standard Form of nth Order Linear Differential Equation:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=f(x) \qquad (*)$$

- If $f(x)=0$, (*) is called homogeneous
- If $f(x)\neq 0$, (*) is called non-homogeneous

Through out this section, we make the following assumptions:

- The coefficient functions $a_n(x), a_{n-1}(x), \ldots\ldots, a_0(x)$ and $f(x)$ are continuous
- $a_n(x)\neq 0$ for every x in the interval

Example:

1) $2e^x y''' + (2\sin x)y'' - 4x^2 y' - 5y = 3$

   linear, 3rd order non-homogeneous DE

2) $xy^{(5)} + 4y^{(4)} + xy''' = 0$

   linear, 5th order homogeneous DE

Remaining part of this section,
presents facts and properties of solutions of

homogeneous linear DE | non-homogeneous linear DE

Derivatives as Differential Operators: The symbol D defined as $D = d/dx$ is called a ***differential operator***. In this notation, the first, second, third and nth order derivatives are defined as:

$$D = d/dx, \quad D^2 = d^2/dx^2, \ldots\ldots, D^n = d^n/dx^n$$

Differential Operator form of nth order homogeneous ODE: In this notation, the nth order homogeneous ODE can be written as

$$a_n(x)D^n y + a_{n-1}(x)D^{n-1} y + \cdots + a_1(x)Dy + a_0(x)y = 0$$

or

$$\left[a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)\right]y = 0$$

Define $L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$ so that the homogeneous ODE becomes

$$Ly = 0$$

Facts about *D* and *L*

The operators D and L are linear, that is,

$$D(\alpha y_1 + \beta y_2) = \alpha D(y_1) + \beta D(y_2)$$

and

$$L(\alpha y_1 + \beta y_2) = \alpha L(y_1) + \beta L(y_2)$$

Example: Homogeneous ODEs in terms of differential operators:

- $y'' + 2y' + y = 0 \Rightarrow D^2 y + 2Dy + y = 0 \Rightarrow \left(D^2 + 2D + 1\right) y = 0$
- $y' + xy = 0 \quad \Rightarrow \quad Dy + xy = 0 \quad \Rightarrow \quad \left(D + x\right) y = 0$

Facts about solution

- Let $Ly = 0$ be an nth order homogeneous ODE.
- If $y_1(x), y_2(x), \cdots y_n(x)$ are n solutions ODE on a given interval, then the linear combination of these solutions given by

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = \sum_{i=1}^{n} c_i y_i(x).$$

  is also a solution of ODE.

Superposition Principle

- If $y(x)$ is a solution of ODE, then $cy(x)$ is also a solution of ODE.
- A homogeneous linear ODE always possesses the trivial solution $y = 0$.

Example: The homogeneous linear ODE $D^2 y + y = 0$ has two solutions given by $y_1 = \cos x$ and $y_2 = \cos x$. Then by superposition principle $y = \alpha \cos x + \beta \sin x$ is also a solution of ODE.

Linearly Independent Solutions: A set of solutions $y_1(x), y_2(x), \cdots, y_n(x)$ is called ***linearly independent*** if the equation

$$c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)=0$$

has only solution $c_1=0, c_2=0, \cdots, c_n=0$.

Otherwise $y_1(x), y_2(x), \cdots, y_n(x)$ are linearly dependent.

Important Facts

A set of two functions $y_1(x)$ and $y_2(x)$ is linearly independent when neither function is a constant multiple of the other on the interval.

For example, the set of functions $y_1(x)=\sin 2x$ and $y_2(x)=\sin x\cos x$ is linearly dependent on $(-\infty,\infty)$ because $y_1(x)$ is a constant multiple of $y_2(x)$ (Recall $\sin 2x = 2\sin x\cos x$).

On the other hand, the set of functions $y_1(x)=x$ and $y_2(x)=|x|$ is linearly independent on $(-\infty,\infty)$ because neither functions is a constant multiple of the other on the interval.

Special Trick to check linear independence

(More practical)

Definition: The Wronskian of functions $y_1(x), y_2(x), \cdots, y_n(x)$ is defined as

$$W(y_1, y_2, \cdots y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

The set of solutions $y_1(x), y_2(x), \cdots, y_n(x)$ of n^th order homogeneous linear differential equation is

(a) ***linearly independent*** **on interval *I* ⇔ the Wronskian $W(y_1, y_2, \ldots, y_n) \neq 0$ for some point $x_0 \in I$.**

(b) ***linearly dependent*** **on interval *I* ⇔ the Wronskian $W(y_1, y_2, \ldots, y_n) = 0$ for some point $x_0 \in I$.**

A linear independent solution of an ODE is called a ***fundamental*** set of solutions.

General Solution: Let $Ly = 0$ be an nth order homogeneous ODE. If $y_1(x), y_2(x), \cdots y_n(x)$ are n linearly independent solutions of the ODE, then the linear combination of these solutions given by

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = \sum_{i=1}^{n} c_i y_i(x).$$

is called the ***general solution*** of the ODE.

Question 17/138: Determine whether $f_1(x)=5, \quad f_2(x)=\cos^2 x, \quad f_3(x)=\sin^2 x$ is linearly independent on the interval $(-\infty,\infty)$.

Question 20/138: Determine whether $f_1(x)=2+x, \quad f_2(x)=2+|x|$ is linearly independent on the interval $(-\infty,\infty)$.

Question 29/138: Verify that $x, \quad x^{-2}, \quad x^{-2}\ln x$ form a fundamental set of solutions of $x^3 y''' + 6x^2 y'' + 4xy' - 4y = 0$ on $(-\infty,\infty)$.