4.1.1 Initial and Boundary Value Problems

Recall the initial value problem (IVP):

***n*th order IVP**:

$$\begin{cases} \text{Solve}: a_n(x)\dfrac{d^n y}{dx^n} + a_{n-1}(x)\dfrac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\dfrac{dy}{dx} + a_0(x)y = f(x) \\ \text{Subject to:} \quad \underbrace{y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}} \end{cases}$$

where $y_0,\ y_1,\ y_2, \ldots,\ y_{n-1}$ are arbitrary real constants.

Initial Conditions

Existence of a Unique Solution: Assume that $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $f(x)$ are continuous on an interval. Then, for any given point in this interval, the solution of IVP exists on the interval and it is unique.

Second Order Boundary Value Problem (BVP):

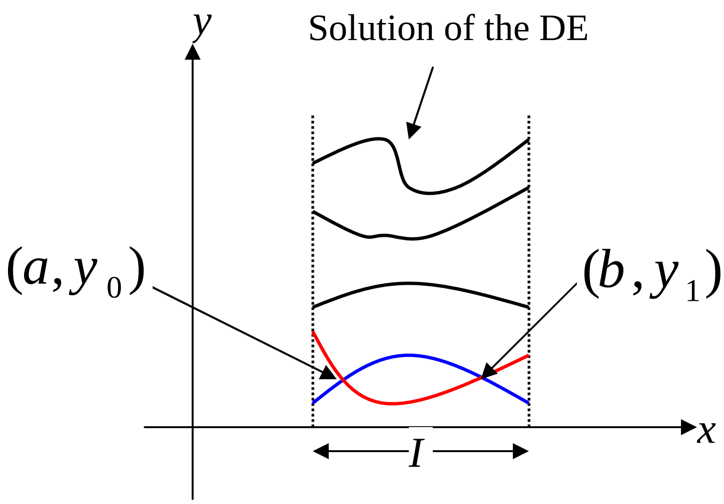
***Second order* BVP**:

$$\begin{cases} \text{Solve}: a_2(x)\dfrac{d^2 y}{dx^2} + a_1(x)\dfrac{dy}{dx} + a_0(x)y = g(x) \\ \text{Subject to:} \quad \underbrace{y(a) = y_0,\ y(b) = y_1} \end{cases}$$

where y_0, y_1 are arbitrary real constants.

Boundary Conditions

A ***solution*** of the BVP is a function that satisfying the DE on some interval I containing a and b whose graph passes through two points (a, y_0) and (b, y_1).



For a second order differential equation, other pairs of boundary conditions could be

$$y'(a) = y_0, \quad y(b) = y_1$$

$$y(a) = y_0, \quad y'(b) = y_1$$

$$y'(a) = y_0, \quad y'(b) = y_1$$

Where y_0 and y_1 are arbitrary constants.

A BVP may have infinitely many, unique or no solutions.

Example: (a) The BVP $y'' + y = 0$ with boundary conditions $y(0) = 0, \; y(2\pi) = 0$ has infinitely many solutions passing through given points.

(b) The BVP $y'' + y = 0$ with boundary conditions $y(0) = 0, \; y(\pi/4) = 0$ has a unique solution $y = 0$

(c) The BVP $y'' + y = 0$ with boundary conditions $y(0) = 0, \; y(2\pi) = 1$ has no solution.

Question 1/137: $y = c_1 e^{x} + c_2 e^{-x}$ is a general solution of IVP $y'' - y = 0$ subject to $y(0) = -1, \quad y'(0) = 6$. Find a solution of the IVP subject to the given condition.

Question 5/137: $y = c_1 + c_2 x^2$ is a two parameter family of solutions of $x^2 y'' - y' = 0$ on the interval $(-\infty, \infty)$. Show that the constants c_1 and c_2 can not be found so that a member of the family satisfies the initial conditions $y(0) = 0, \quad y'(0) = 1$.

Question 9/138: Find an interval centered about $x = 0$ for which $(x - 2)y'' + 3y = x; \; y(0) = 0, \quad y'(0) = 1$.

Question 12/138: Use the family $y = c_1 + c_2 x^2$ to find the solution of $xy'' - y' = 0$ that satisfies the boundary conditions $y(0) = 1, \quad y'(1) = 6$.