3.1 Linear Models

Objectives: We will see how the linear ODEs can be used to represent real world problems, for example, population growth and decay problems, Newton's law of cooling problems, and mixture problems.

Polulation Growth Model: The rate at which the population of a country grows at a certain time is proportional to the total population of the country at that time.

In other words, if $P(t)$ denotes the total population at time t, then

$\frac{dP}{dt} \propto P$ or $\frac{dP}{dt} = kP$, where k is a constant of proportionality.

$\frac{dP}{dt}$ denotes the rate of change of P with respect to time t.

The intial value problem, $\frac{dP}{dt} = kP$ with $P(t_0) = P_0$, represents ***exponential growth*** if "k" is positive or ***exponential decay*** if "k" is negative.

Why Exponential Growth or Decay: If we solve the above IVP by using separable variable method, then we get the solution $P = P_0 e^{kt}$. The exponential function e^{kt} increases as t increases for $t > 0$ and decreases as t increases for $t < 0$.

Question 4/98: The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t. After 3 hours it is observed that 400 bacteria are present. After 10 hours 2000 bacteria are present. What was the initial number of bacteria?

Newton's Law of Cooling / Warming: Newton's law of cooling states that the rate of change of the temperature of an object is proportional to the difference between the temperature of the body and the temperature of the surrounding medium also called ambient temperature.

- The temperature of a body at time t: $T(t)$
- The temperature of the surrounding medium at time t: T_m
- The rate at which the temperature of the body changes: dT/dt
- Newton's of cooling is: $\frac{dT}{dt} \propto T - T_m$ or $\frac{dT}{dt} = k(T - T_m)$, where k is a constant of proportionality.

Question 17/99: A thermometer reading 70° F is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads 110° F after ½ minute and 145° F after 1 minute. How hot is the oven?

Mixture Problem:

- Consider a container containing some initial amount of brine.
- Suppose we pump in another brine solution (with some given concentration of salt per pound per gallon) at a constant rate per unit time.
- Now mix the solutions thoroughly and let the solution drain out at the same amount at which the solution is entering the container.
- Let $S(t)$ be amount of salt in a container at any time t. Then the rate at which $S(t)$ changes is the net rate:

$$\frac{dS(t)}{dt} = \begin{pmatrix} \text{input rate} \\ \text{of salt} \end{pmatrix} - \begin{pmatrix} \text{out rate} \\ \text{of salt} \end{pmatrix} = R_{in} - R_{out}$$

R_{in} = input rate of salt

= concentration of salt in inflow × input rate of brine

R_{out} = output rate of salt

= concentration of salt in outflow × output rate of brine

Question 25/100: A large tank is partially filed with 100 gallons of fluid in which 10 pounds of salt is dissolved. Brine containing ½ pound of salt per gallon is pumped into the tank at a rate of 6 gal/min. The well-mixed solution is then pumped out at a slower rate of 4 gal/min. Find the number of pounds of salt in the tank after 30 minutes.