2.5 Solutions by Substitutions

Objectives:

- To learn technique of transforming an ODE equation to a separable equation
- To know homogeneous functions and homogeneous ODE
- To recognize and solve homogeneous first order ODEs
- To recognize and solve Bernoulli equation.

A differential equation of the form

$$\frac{dy}{dx} = f(ax + by + c); \quad b \neq 0 \qquad \{\text{what happens for } b = 0\,?\}$$

can always be ***reduced to a separable equation*** by the

substitution $\quad u = ax + by + c$

where a, b, and c are constants.

Question 24/78: Solve the ODE $\frac{dy}{dx} = \frac{1 - x - y}{x + y}$ by an appropriate substitution.

Question 29/78: Solve the IVP $\frac{dy}{dx} = \cos(x + y),\ y(0) = \pi / 4$ by an appropriate substitution.

Homogeneous Function: If $f(tx,ty)=t^n f(x,y)$ for some real number n, then $f(x,y)$ is called a ***homogeneous function*** of degree n.

For example, $f(x,y)=x^3+y^3$ is a homogeneous function of degree 3, because $f(tx,ty)=t^3 f(x,y)$. The function $f(x,y)=x^3+y^3+1$ is not homogeneous.

Homogeneous ODE: A first order DE in differential form

$$M(x,y)dx+N(x,y)dy=0$$

is said to be ***homogeneous*** if both coefficient functions M and N are homogeneous functions of the *same* degree.

If M and N are homogeneous functions of degree n, then we can always write

$M(x,y)=x^n M(1,u)$ and $N(x,y)=x^n N(1,u)$ where $u=y/x$

and

$M(x,y)=y^n M(v,1)$ and $N(x,y)=y^n N(v,1)$ where $v=x/y$.

The homogeneous ODE is solved by substituting $y=ux$ or $x=vy$.

In particular, a homogeneous 1st order ODE is the one that can be written in the form

$$\boxed{\frac{dy}{dx}=F\left(\frac{y}{x}\right)} \qquad \textbf{(*)}$$

Examples:

a) $\dfrac{dy}{dx}=\dfrac{2x^2y}{3x^3+y^3}$

Why?

b) $x\dfrac{dy}{dx}=y\left(\ln y-\ln x\right)$

Method of solution

Main idea: Given $\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$

The substitution $\boxed{u = \dfrac{y}{x}}$ **(or** $y = ux$ **) and hence** $\boxed{\dfrac{dy}{dx} = u + x\dfrac{du}{dx}}$ **transforms the ODE into a separable equation.**

Steps for finding the solution:

Step 1: **Check** if **homogeneous.** Try to write in the form $\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$ **(*)** **{**If homogeneous, proceed as follows**}**

Step 2: **Substitute** $\boxed{u = \dfrac{y}{x}}$ **(or** $y = ux$**) and** $\boxed{\dfrac{dy}{dx} = u + x\dfrac{du}{dx}}$ in (*). This transforms it into separable equation.

Step 3: **Solve** the **separable equation**.

Step 4: Use the substitution $\boxed{u = \dfrac{y}{x}}$ to get the **general solution in** the variables ***y* and *x*.**

Step 6: In case of **IVP**, find the **particular solution**.

Question 9/78: Solve the ODE $-ydx + \left(x + \sqrt{xy}\right)dy = 0$ by appropriate substitution.

Question 14/78: Solve the IVP $ydx + x(\ln x - \ln y - 1)dy = 0,\ \ y(1) = e$ by appropriate substitution.

Bernoulli Equation: A first order ODE of the form

$$\frac{dy}{dx} + P(x) \cdot y = Q(x) \cdot y^n \qquad (*)$$

is called ***Bernoulli equation***.

Note that for $n = 0$ and $n = 1$, equation (*) is linear.

Method of solution

The substitution $u = y^{1-n}$ to transforms the Bernoulli equation into a linear equation as follows:

- $\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y^n}{(1-n)}\frac{du}{dx}$
- $\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$

Question 20/78: Solve ODE $3(1+t^2)\frac{dy}{dt} = 2ty(y^3 - 1)$ by an appropriate substitution.

Question 22/78: Solve IVP $y^{1/2}\frac{dy}{dx} + y^{3/2} = 1, \ \ y(0) = 4$ by an appropriate substitution.