2.4 Exact Ordinary Differential Equations

Objectives:

- To know the meaning of an exact equation
- To recognize when a first order ODE is exact
- How to solve an exact equation.

Explanation and Definition of Exact equations:

- Consider the following function which has continuous first derivatives in a region of xy-plane

$$z = f(x, y) \qquad (1)$$

- Take differential of (1)

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad (2)$$

- Assume that $z = c$ (a constant), then (2) becomes

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \qquad (3)$$

- Call $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$ so that (3) can be written as

$$\boxed{M(x, y)dx + N(x, y)dy = 0} \qquad (4)$$

An first order ODE of the form $M(x, y)dx + N(x, y)dy = 0$ is called ***exact*** if the expression $M(x, y)dx + N(x, y)dy$ is exact differential of some function $f(x, y)$; that is

$$df(x, y) = M(x, y)dx + N(x, y)dy$$

Criterion for an Exact Differential: Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in a rectangular region $R = \{(x,y) : a < x < b,\ c < y < d\}$. Then a necessary and sufficient condition that $M(x,y)dx + N(x,y)dy$ be an exact differential is

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

In other words, a necessary and sufficient condition that the equation $M(x,y)dx + N(x,y)dy = 0$ be an exact is

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}.$$

Reason behind an Exact ODE:

- ODE Exact if: $\dfrac{\partial M(x,y)}{\partial y} = \dfrac{\partial N(x,y)}{\partial x}$
- Observe that: $M(x,y) = \dfrac{\partial f}{\partial x}$ and $N(x,y) = \dfrac{\partial f}{\partial y}$
- Thus

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

The main idea to solve exact ODE $M(x,y)dx + N(x,y)dy = 0$:

❖ Find a function $f(x,y)$ such that $df(x,y) = M(x,y)dx + N(x,y)dy$

that is, $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$

❖ Then $f(x,y) = c$ gives solution.

Steps to Solve an Exact ODE:

- **Step 1.** **Re-write** the ODE in the form

$$M(x,y)dx + N(x,y)dy = 0$$

- **Step 2.** **Identify** $M(x,y)$ and $N(x,y)$

- **Step 3.** **Check exactness:** $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$

If exact proceed further

- **Step 4.** To find the function $f(x,y)$ so that $\frac{\partial f}{\partial x} = M(x,y)$, we

integrate w.r.t. x

$$f(x,y) = \int M(x,y)dx + g(y) \qquad (*)$$

where $g(y)$ is the integration constant (a function of y)

- **Step 5.** Since we also want $f(x,y)$ to satisfy $\frac{\partial f}{\partial y} = N(x,y)$; so we differentiate $f(x,y)$, obtained in (*), w.r.t. y to get

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial\left(\int \frac{\partial f}{\partial x}\, dx\right)}{\partial y} + \frac{\partial g(y)}{\partial y}$$

  and set it equal to $N(x,y)$.

- **Step 6.** Plug the integration constant $g(y)$ in $f(x,y)$, obtain from (*), to have the final form of the function $f(x,y)$.
- **Step 7.** Get the ***general solution*** by setting $f(x,y) = c$.
- **Step 8.** In case of IVP, find the ***particular solution***.

Question 9/73: Is $\left(x - y^3 + y^2 \sin x\right)dx = \left(3xy^2 + 2y\cos x\right)dy$ exact? If so, solve it.

Question 14/73: Is $\left(1 - \frac{3}{y} + x\right)\frac{dy}{dx} + y = \frac{3}{x} - 1$ exact? If so, solve it.

Question 26/73: Solve IVP

$$\left(\frac{1}{1+y^2} + \cos x - 2xy\right)\frac{dy}{dx} = y\left(y + \sin x\right), \quad y(0) = 1.$$

Integrating Factor and Non-Exact ODE: Some time a non-exact ODE can be made exact by using integrating factor. This method consists of the following steps:

- **Step 1**. **Re-write** the ODE in the form

$$M(x,y)dx + N(x,y)dy = 0$$

- **Step 2**. **Check non-exactness:** $\dfrac{\partial M(x,y)}{\partial y} \neq \dfrac{\partial N(x,y)}{\partial x}$

- **Step 3**. **Multiply ODE by** $\mu(x)$ **(called IF)**:

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

- **Step 4**. Use the fact now that:

$$\begin{cases} \dfrac{\partial \mu M(x,y)}{\partial y} = \dfrac{\partial \mu N(x,y)}{\partial x} \\ \mu M_y = N\dfrac{d\mu}{dx} + \mu N_x \\ \dfrac{d\mu}{dx} = \dfrac{M_y - N_x}{N}\mu \end{cases}$$

- Consider two possibilities for $\dfrac{M_y - N_x}{N} = \begin{cases} f(x) \\ \neq f(x) \end{cases}$

- In first case, $\dfrac{M_y - N_x}{N} = f(x)$, gives rise to an integrating factor $\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$

- Check that $\mu(x)M(x,y)dx+\mu(x)N(x,y)dy=0$ is now exact, that is, $\frac{\partial\mu M(x,y)}{\partial y}=\frac{\partial\mu N(x,y)}{\partial x}$

- In the second case, $\frac{M_y-N_x}{N}\neq f(x)$, the $\mu(x)$ gives nothing.

- **Step 5**. **Multiply ODE by $\mu(y)$ (called IF):**

$$\mu(x)M(x,y)dx+\mu(x)N(x,y)dy=0$$

- **Step 6**. Use the fact now that:

$$\begin{cases}\frac{\partial\mu M(x,y)}{\partial y}=\frac{\partial\mu N(x,y)}{\partial x}\\ \mu N_x=M\frac{d\mu}{dy}+\mu M_y\\ \frac{d\mu}{dy}=\frac{N_x-M_y}{M}\mu\end{cases}$$

- Consider two possibilities for $\frac{N_y-M_y}{M}=\begin{cases}f(y)\\ \neq f(y)\end{cases}$

- In first case, $\frac{N_x-M_y}{M}=f(y)$, gives rise to an integrating factor $\mu(y)=e^{\int\frac{N_x-M_y}{M}dy}$

- Check that $\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0$ is now exact, that is, $\frac{\partial \mu M(x,y)}{\partial y} = \frac{\partial \mu N(x,y)}{\partial x}$
- In the second case, $\frac{N_x - M_y}{M} \neq f(y)$, the $\mu(y)$ gives nothing.

> The integrating factors (IF) can be in general any function of 'x' and 'y'. However, we have restricted ourselves to special cases for this course.

Question 30/73: Verify that the ODE $\left(x^2 + 2xy - y^2\right)dx + \left(y^2 + 2xy - x^2\right)dy = 0$ is not exact. Multiply the ODE with integrating factor $\mu(x,y) = \left(x + y\right)^{-2}$ and verify that the resulting ODE is now exact. Solve the exact ODE.

Question 36/73: Solve the ODE $\left(y^2 + xy^3\right)dx + \left(5y^2 - xy + y^3 \sin y\right)dy = 0$ by finding an appropriate integrating factor.

Integrating Factor and Non-Exact ODE: Some time a non-exact ODE can be made exact by using integrating factor. This method consists of the following steps:

- **Step 1**. **Re-write** the ODE in the form

$$M(x,y)dx + N(x,y)dy = 0$$

- **Step 2**. **Check non-exactness:** $\dfrac{\partial M(x,y)}{\partial y} \neq \dfrac{\partial N(x,y)}{\partial x}$

- **Step 3.**
  - **If** $\dfrac{M_y - N_x}{N} = f(x)$ (a function of x), then the Integrating Factor (IF) is $\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$.
  - **If** $\dfrac{M_y - N_x}{N} \neq f(x)$ (not a function of x), then the $\mu(x)$ gives nothing. Go to Step 5.

- **Step 4**. **Multiply ODE by** $\mu(x)$ **(called IF)**:

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

  - Check that $\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$ is now exact, that is, $\dfrac{\partial \mu M(x,y)}{\partial y} = \dfrac{\partial \mu N(x,y)}{\partial x}$

- **Step 5**.

  - **If** $\frac{N_x - M_y}{M} = f(y)$ (a function of y), then the Integrating Factor (IF) is $\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$.

  - **If** $\frac{N_x - M_y}{M} \neq f(y)$ (not a function of y), then the $\mu(y)$ gives nothing. Go to Step 3.

- **Step 6**. **Multiply ODE by $\mu(y)$ (called IF)**:

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

  - Check that $\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0$ is now exact, that is, $\frac{\partial \mu M(x,y)}{\partial y} = \frac{\partial \mu N(x,y)}{\partial x}$

- **Step 7**. Solve the exact ODE given in Step 4 or 6 by using the previous method.