2.2 Separable Variables

Definition: A first order ODE $\frac{dy}{dx} = f(x,y)$ is called ***separable*** if it can be written in the form

$$\frac{dy}{dx} = g(x)\cdot h(y) \qquad \text{or} \qquad N(y)dy = M(x)dx$$

That is, the x-terms and y-terms can be separated on two sides of the equation

Examples:

1. $\frac{dy}{dx} = xy^2e^{3x}$ is separable.
2. $\frac{dy}{dx} = y + \cos x$ is not separable.

Why?

Which one of these is linear ODE

Aim: To learn to find solutions of separable differential equations

Kinds of Solutions of $\frac{dy}{dx} = f(x,y)$:

- **Explicit solutions:** of the form $y = f(x)$
- **Implicit solutions:** of the form $F(x,y) = 0$
- **Singular or constant solutions:** the solutions $y = k$ obtained by solving $f(x,y) = 0$.

Note: ***You can obviously see that constant solutions will not always exist.***

A DE may some time have additional solution that can not be obtained from its general solution. Such solution is called ***singular solution***.

Methods of Solving Separable Differential Equations:

(Main idea: Separate and integrate)

Question: Given separable equation $\frac{dy}{dx} = f(x, y)$. To find solution $y(x)$.

Step 1: Separate the equation to put in the form $N(y)dy = M(x)dx$

Step 2: Integrate both sides to get a general solution.

Step 3(a): Solve for $y(x)$ to get a general explicit solution.

3(b): If explicit solution is not possible, write the implicit solution.

Step 4: (If given) use initial conditions (e.g. in case of IVP) to get particular solution.

Step 5: Check your solution by taking derivatives & putting back in equation.

Further, be careful about the following:

Even if you have done all the above steps, you may not get correct or all solutions because of the following possible problems.

Caution 1: In case of IVP, be sure that you have chosen the correct solution.

Caution 2: If there are any singular and constant solutions, they may or may not be contained in general solution.

In case they are not, be sure to write these solutions as well.

Caution 3: Be sure that the arbitrary constant is introduced immediately after the integration and not later.

Question 9/54: Solve $y \ln x \dfrac{dy}{dx} = \left(\dfrac{y+1}{x} \right)^2$.

Question 13/54: Solve $(e^y + 1)^2 e^{-y} dx + (e^x + 1)^3 e^{-x} dy = 0$.

Question 19/54: Solve $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$.

Question 25/54: Find an implicit and an explicit solution of the IVP $x^2 \dfrac{dy}{dx} = y - xy$ subject to $y(-1) = -1$.

Question 30(c)/55: Find a solution of $x \dfrac{dy}{dx} = y^2 - y$ that passes through the point $\left(\dfrac{1}{2}, \dfrac{1}{2} \right)$.

Question 31/55: (Also Question 21): Find a singular solution of the DE $\dfrac{dy}{dx} = x\sqrt{1 - y^2}$.