2.1 Solution Curves With out a Solution

Objectives:

- **What are direction fields?**
- **Why do we need to learn to plot direction fields?**
- **Plotting direction fields using MATLAB**
- **Example**
- **Direction fields and solutions on same graph window**
- **To study a new type of first order ODEs in which independent variable does not explicitly depend on independent variable "called autonomous first order ODEs"**

Direction Field: Consider the first order ODE

$$\frac{dy}{dx} = f(x, y) \qquad (*)$$

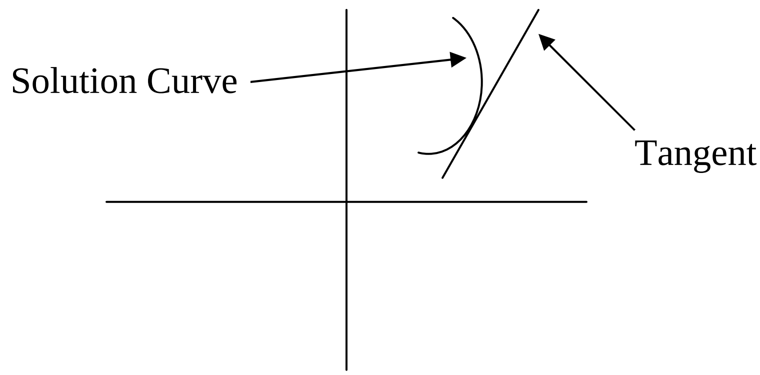
- Let $y(x)$ be a solution of $\frac{dy}{dx} = f(x, y)$ whose graph is a **solution curve** of ODE.
- If (x_0, y_0) is a point on solution curve, then the **slope of the tangent line to the solution curve** at (x_0, y_0) is given by

$$f(x_0, y_0) \qquad \left\{ \text{Since } \left.\frac{dy}{dx}\right|_{(x_0, y_0)} = f(x_0, y_0) \right\}$$

Key point: we can get information about slope, of the solution curve, at any point directly from Eq. (*) without solving it.

- Hence, **we can draw small line segments with slope** $f(x_i, y_i)$ **at any desired point** (x_i, y_i). {***Usually we take a reasonably good collection of points to draw these lines***}.

- **The set of all these line segments is called a *direction field* (or slope field);** because at each point (x_i, y_i) it gives direction (or slope) of the tangent line, to the solution curve, at (x_i, y_i).



Why do we need to learn to plot direction fields?

- Many time it will be impossible or too difficult to solve ODE of the type given by Eq. (*).
- But by plotting direction fields, you can get a good (geometric) idea about the solution and its properties.
- It is extremely useful because
  - Plotting direction fields is quite easy (you don't have to solve ODE, you use it as it is).
  - It gives a pretty good idea of how the solution should like (because a solution curve is tangent to "small line segments" that are plotted)
  - You can use it as an aid to verify your approximate or numerical solutions.

Plotting direction fields in MATLAB:

a) Aim

To plot direction fields of

$$\frac{dy}{dx} = f(x, y) \qquad (*)$$

b) What do we need to do

- Choose points where we want to draw slope fields
- Find the slopes at these points directly from Eq. (*).
- Draw the slope fields

c) We need two main commands of MATLAB

- **meshgrid**

  >>[*x*,*y*]=meshgrid(*a*:*k*:*b*,*c*:*j*:*d*);

  creates a set of points (x,y) where

  i. 'x' lies between 'a' & 'b', incremented by 'k'.

  ii. 'y' lies between 'c' & 'd', incremented by 'j'.

> e.g. >>[x,y]=meshgrid(1:0.5:2,0:1:2);
>
> creates the set of nine points
>
> (1,0), (1,1), (1,2),
>
> (1.5,0), (1.5,1), (1.5,2),
>
> (2,0), (2,1), (2,2).

- **quiver**

  >> quiver(*a*,*b*,*x*,*y*)

  begins at the point (a,b) and plots the vector $v = (x, y)$.

Example:

a) Aim

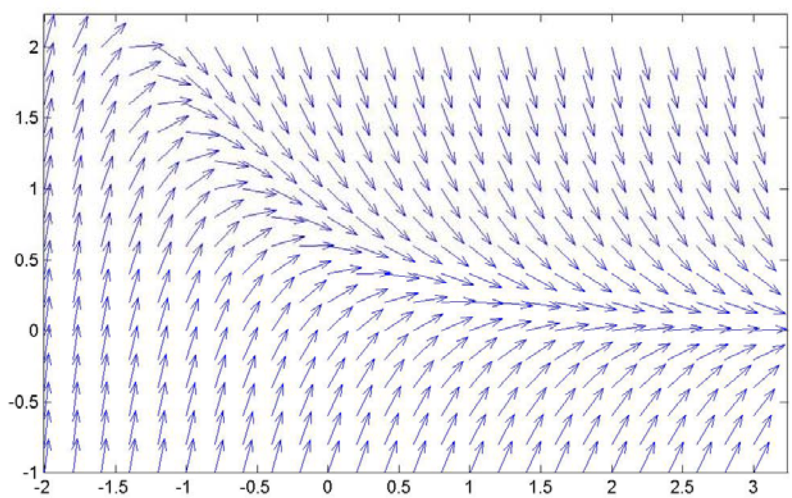
To plot direction fields of $\dfrac{dy}{dx} = e^{-x} - 2y$

on the rectangle $-2 \le x \le 3$, $-1 \le y \le 2$.

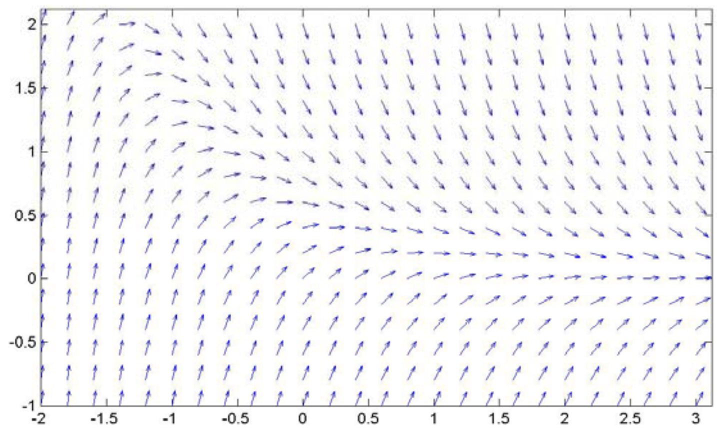
b) Can be done using following MATLAB commands

```
>>[x,y]=meshgrid(-2:0.2:3,-1:0.2:2);
>>dy=exp(-x)-2*y;
>>dx=ones(size(dy));
>>dyu=dy./sqrt(dx.^2+dy.^2);
>>dxu=dx./sqrt(dx.^2+dy.^2);
>>quiver(x,y,dxu,dyu);
```

These commands generate the following:

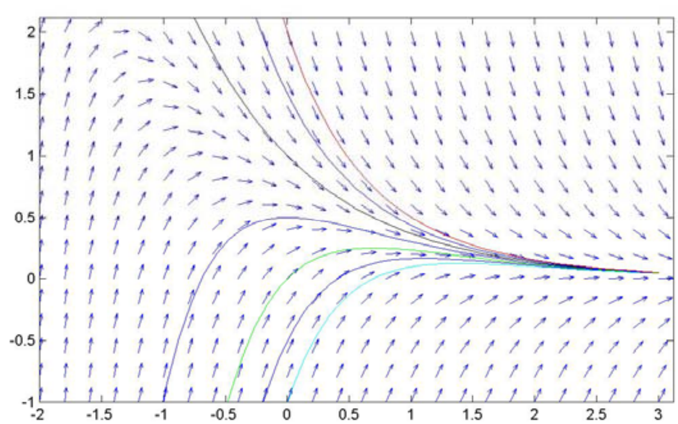


We can do better using the command **>>quiver(x,y,dxu,dyu,0.5); (why)** which gives



Direction Fields & Solutions on same Graph Window:

- Once you have plotted the direction field you can plot the solution curves (if you know the general solution) or your numerical solution on the same graph window using "**hold on**" and "**hold off**" as explained earlier.
- The general solution of the ODE in our example is $y = e^{-x} + Ce^{-2x}$. The following figure shows direction fields with some of these solutions.



Autonomous ODE: An ODE in which independent variable does not appear explicitly is called an ***autonomous ODE***. Such ODE is generally written as $\frac{dy}{dx} = f(y)$ and $f(y, y') = 0$.

Examples:

1. $\frac{dy}{dx} = 1 + y^4$
2. $\frac{dA(t)}{dt} = kA(t)$

Critical / Equilibrium Points: The points, at which $f(y)$ in the ODE given by $\frac{dy}{dx} = f(y)$ is zero, are called ***critical*** or ***equilibrium prints***.

In particular, "c" is called a critical point if $f(c) = 0$.

Solution at Critical Points: When "c" is a critical point of the ODE $\frac{dy}{dx} = f(y)$, then $y(x) = c$ is a constant solution of the autonomous ODE.

This constant solution of the autonomous ODE is called ***equilibrium solution***.

Question 19/48: Find critical points of the first order autonomous ODE

$$\frac{dy}{dx} = y - y^3.$$

Solution: The ODE has three critical points because $f(y) = y - y^3 = 0$ at $y = 0$ and $y = \pm 1$.

Question 22/48: Find critical points of the first order autonomous ODE

$$\frac{dy}{dx} = y^2 - y^3.$$

Solution: The ODE has two critical points because

$f(y) = y^2 - y^3 = 0 \quad \Leftrightarrow \quad y^2(1 - y) = 0$ at $y = 0$ and $y = 1$.