1.2 Initial-Value Problems

Initial-Value Problem: An ***initial-value problem*** (IVP) consists of a differential equation and initial condition(s).

***n*th order IVP**:

$$\begin{cases} \text{Solve:} & \dfrac{d^n y}{dx^n} = f(x, y, y', \ldots\ldots, y^{(n-1)}) & (1) \\ \text{Subject to:} & \underbrace{y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots\ldots,\ y^{(n-1)}(x_0) = y_{n-1}} & (2) \end{cases}$$

Initial Conditions

where $y_0,\ y_1,\ y_2, \ldots\ldots,\ y_{n-1}$ are arbitrary real constants.

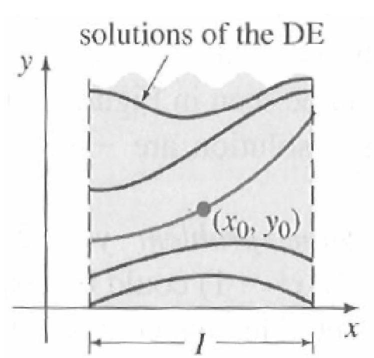
In particular,

- **First order IVP**:

$$\begin{cases} \text{Solve:} & \dfrac{dy}{dx} = f(x, y) & (3) \\ \text{Subject to:} & y(x_0) = y_0 & (4) \end{cases}$$

Initial Condition

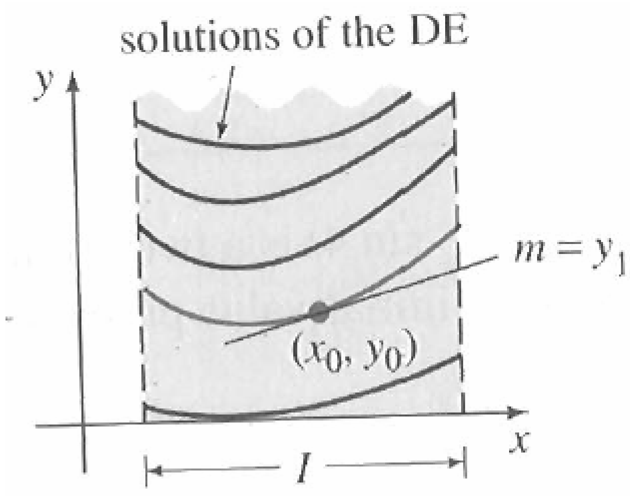
For this IVP, we are seeking a solution $y(x)$ of the DE $y' = f(x, y)$ on an interval I containing x_0 so that its graph passes through the specified point (x_0, y_0).



- **Second order IVP**:

Initial Conditions

$$\begin{cases} \text{Solve}: & \dfrac{d^2y}{dx^2} = f(x, y, y') & (5) \\ \text{Subject to:} & y(x_0) = y_0,\ y'(x_0) = y_1 & (6) \end{cases}$$



Solution of IVP: To find a function $y(x)$ that satisfies ODE given by Equation (7) and also the initial conditions given by (8).

Example: Given that $y = ce^x$ gives solutions of the ODE $\dfrac{dy}{dx} = y$. Find the solution to the following IVP.

$$\frac{dy}{dx} = y; \qquad y(0) = 3.$$

Solution:

Using initial condition in $y = ce^x$ implies

$$3 = ce^0 \qquad \text{or} \qquad c = 3.$$

Hence, $y = 3e^x$ is the solution of IVP.

Existence of Unique Solution (Theorem 1.1):

Let $R=\{(x,y): a\le x\le b,\ \ c\le y\le d\}$ be a region that contain the point (x_0, y_0). If $f(x,y)$ and $\partial f/\partial y$ are continuous on R, then there exists a unique solution $y(x)$ of the problem on the interval $I_0=(x_0-h, x_0+h)$ where $h>0$. This unique solution is what we call the solution of the IVP.

Question 1/16: $y=\dfrac{1}{(1+c_1e^{-x})}$ is a one-parameter family of solutions of first order DE $y'=y-y^2$. Find a solution of the IVP consisting of this DE and the initial condition $y(0)=1/4$.

Question 4/16: $y=\dfrac{1}{(x^2+c)}$ is a one-parameter family of solutions of first order DE $y'+2xy^2=0$. Find a solution of the IVP consisting of this DE subject to $y(-2)=1/2$. Also, give largest interval over which the solution is defined.

Question 8/17: $x=c_1\cos t+c_2\sin t$ is a two-parameter family of solutions of second order DE $x''+x=0$. Find a solution of the IVP consisting of this DE subject to $x(\pi/2)=0$ and $x'(\pi/2)=1$.

Question 13/17: $y = c_1 e^{x} + c_2 e^{-x}$ is a two-parameter family of solutions of second order DE $y'' + y = 0$. Find a solution of the IVP consisting of this DE and initial conditions $y(-1) = 5$ and $y'(-1) = -5$.

Question 24/17: Determine the region on the xy-plane for which $(y - x)y' = y + x$ would have a unique solution whose graph passes through a point (x_0, y_0) in the region.

Question 28/17: Does Theorem 1.1 guarantees that ODE $y' = \sqrt{y^2 - 9}$ has a unique solution through the point $(-1,1)$.