1.1 Definitions and Terminology

Differential Equation (DE)

An equation that contains one or several derivatives of one or more dependent variables with respect to one or more independent variables is called a DE.

Ordinary Differential Equation (ODE)

An equation that contains one or several ordinary derivatives of one or more dependent variables with respect to single independent variable is called an ODE.

Partial Differential Equation (PDE)

An equation that contains one or several partial derivatives of one or more dependent variables with respect to two or more independent variables is called a PDE.

This course deals with the study of ODE.

Through out this course, ordinary derivatives will be used either Leibniz notations dy/dx, d^2y/dx^2, d^3y/dx^3,....., d^ny/dx^n or the prime notations $y', y'', y''', y^{(4)}, \ldots, y^{(n)}$.

Examples of ODEs:

1. $\dfrac{dy}{dx} = \sin x$
2. $\dfrac{d^2y}{dx^2} + 4y = 0$
3. $\dfrac{dy}{dt} + y^2 \sin t = 3e^{t^2}$
4. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 6y = 0$
5. An ODE can contain more than one dependent variable

   $$\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

6. $\dfrac{d^2x}{dt^2} + 16x = 0$

   unknown function or dependent variable (x)

   independent variable (t)

> Can you recognize the dependent and independent variables in these equations?

Order of an ODE: The order of the highest derivative in an ODE is called the ***order*** of the ODE.

Examples:

1. second order ($\frac{d^2y}{dx^2}$), first order ($\frac{dy}{dx}$)

   $$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

   Second order ODE

2. $\dfrac{d^2x}{dt^2} + 16x = 0$

   Second order ODE

3. $y' + xy + \sin x = 2$ — First order ODE

4. $y^5 y''' + (y')^2 + e^x y = \tan x$ — Third order ODE

Symbolic Representation of ODEs:

- The most ***general form*** of an nth order ODE is

$F(x, y, y') = 0$ — 1^{st} order ODE

$F(x, y, y', y'') = 0$ — 2^{nd} order ODE

$\vdots$

$F(x, y, y', y'', \dots, y^{(n)}) = 0$ — n^{th} order ODE

- An nth order ODE is said to be in ***normal form*** if it can be written as

$$\frac{d^n y}{dx^n} = y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)})$$

i.e. can be solved explicitly for the highest derivative

a) ***Normal form*** of $4xy' + y - x = 0$ is

$$y' = \frac{x - y}{4x}.$$

b) The equation $(y')^2 + y^2 = 1$ can not be written in normal form.

For simplicity, we will be dealing with ODE's that can be written in normal form.

Linear and Non-Linear ODE's: A differential equation of order "n" is called ***linear*** if it is of the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + y = f(x), \quad a_n(x) \neq 0.$$

In other words, an ODE $F(x, y, y', y'', \dots, y^{(n)}) = 0$ is said to be ***linear*** if F is a linear function of $y, y', \dots, y^{(n)}$.

Otherwise ODE is said to be ***non-linear***.

Note: It does not matter whether or not F is linear in x.

Examples:

a) $y'' + xy = 0$; second order; linear

b) $y' + y^2 = 0$; first order; non-linear

c) $2yy' + y'' = 0$; second order; non-linear

d) $y'' + \sin y = 0$; second order; non-linear

e) $y'' + \sin x = 0$; second order; linear

f) $y'' + y \sin x = 0$; second order; linear

Why?

g) $dy/dx = \sin y$; first order; non-linear

h) $y\dfrac{dy}{dx} = 1$; first order; non-linear

i) $\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$; second order; non-linear

Where & how do the ODEs arise?

ODE	Situation represented by the ODE
$\dfrac{dy(t)}{dt} = k\, y(t)$ $\Rightarrow\; y(t) = c\, e^{kt}$	Population growth rate (dy/dt) proportional to Population present
$\dfrac{d^2y(t)}{dt^2} = k\, g$ $\Rightarrow\; y(t) = (k/2)gt^2 + v_0 t + y_0$	Acceleration of a falling object is proportional to "g"

Solution of an ODE: A ***solution*** of an nth-order ODE $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ is a function ϕ, defined on an interval I, that possesses at least n derivatives that are continuous on I and for which

$$F(x, \phi(x), \phi'(x), \phi''(x), \ldots, \phi^{(n)}(x)) = 0 \qquad \text{for all } x \in I.$$

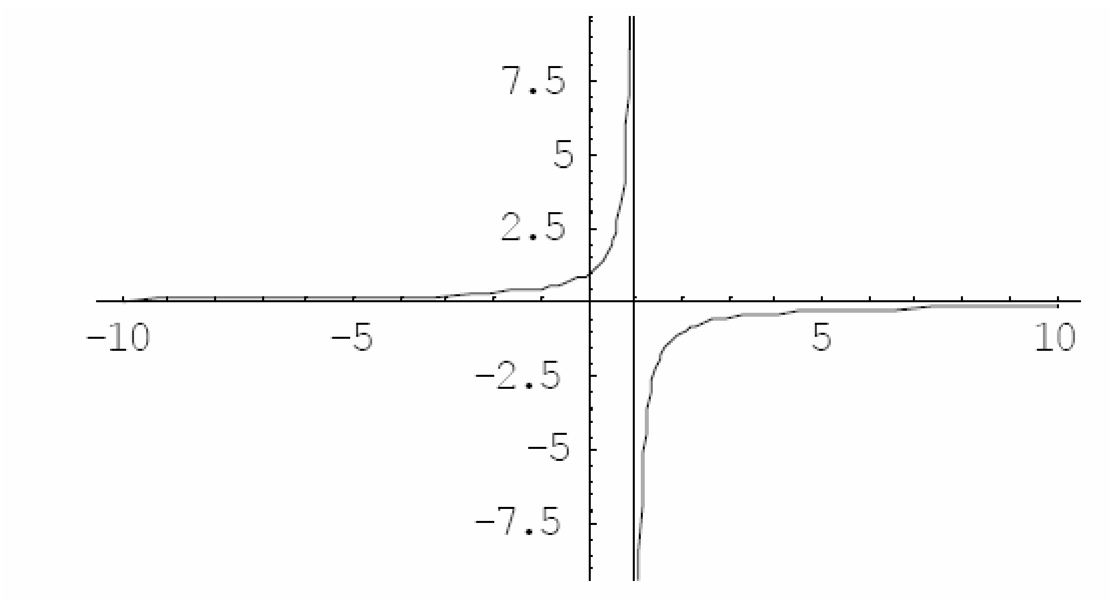
For the sake of convenience, we use alternative symbol $y(x)$ for the solution.

The interval I is called the ***interval of definition***, the ***interval of existence***, the ***interval of validity***, or the ***domain of the solution***.

The graph of a solution ϕ of an ODE is called a ***solution curve***.

Verification of a Solution: To verify the given function is a solution of an ODE, we see that, after substitution, whether each side of the ODE is the same for every x in the interval.

Example. Verify that $y(x) = (1-x)^{-1}$ is a solution of $dy(x)/dx = y^2$. Plot the graph of $\phi(x)$ for -10 < x < 10.



Verification: Since $dy(x)/dx = (1-x)^{-2}$,

LHS of the ODE = $dy/dx = (1-x)^{-2} = y^2$ = RHS. It is a continuous solution on $(-\infty,1) \cup (1,\infty)$ but has vertical asymptote at $x = 1$.

Explicit Solution: A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an ***explicit solution***.

Example: $y = \frac{1}{16}x^4$ and $y = xe^x$ are explicit solutions of $dy/dx = xy^{1/2}$ and $y'' - 2y' + y = 0$, respectively.

Question 14/10: Verify that $y = -(\cos x)\ln(\sec x + \tan x)$ is an explicit solution of the function $y'' + y = \tan x$.

(Use: $\frac{d}{dx}\ln(\sec x + \tan x) = \sec x$)

Implicit Solution: A relation $f(x, y) = 0$ is said to be an ***implicit solution*** of an ODE on an interval I if there exists al least one function ϕ that satisfies the relation as well as the ODE on the interval I.

Question 19/10: The relation $f(X, t) = \ln\left(\frac{2X-1}{X-1}\right) - t = 0$ is an implicit solution of the ODE $\frac{dX}{dt} = (X-1)(1-2X)$ on the interval $(-\infty, \ln 2)$ or $(\ln 2, \infty)$.

Question 20/10: The relation $f(x, y) = -2x^2y + y^2 - 1 = 0$ is an implicit solution of the ODE $2xydx + (x^2 - y)dy = 0$ on the interval $(-\infty, \infty)$.

Solution:

Step 0: Why implicit solutions?

Using the quadratic formula to solve $y^2 - 2x^2y - 1 = 0$ for y, We get $y = \left(2x^2 \pm \sqrt{4x^4 + 4}\right)/2 = x^2 \pm \sqrt{x^4 + 1}$. Thus both $y_1 = x^2 + \sqrt{x^4 + 1}$ and $y_1 = x^2 - \sqrt{x^4 + 1}$ satisfy relation as well as ODE on the interval $(-\infty, \infty)$.

Sept 1: Implicit differentiation w.r.t. x gives

$$-2x^2\frac{dy}{dx}-4xy+2y\frac{dy}{dx}=0 \Rightarrow -x^2dy-xydx+ydy=0$$

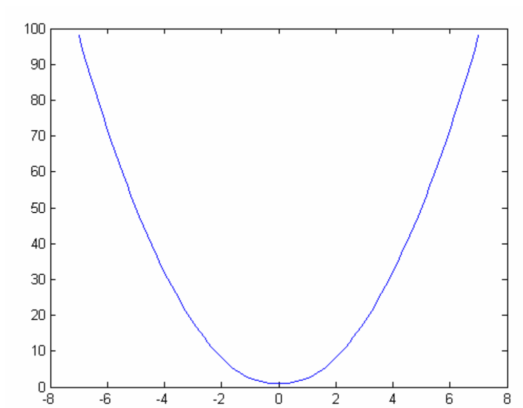
Step 2: Simplify above expression to get:

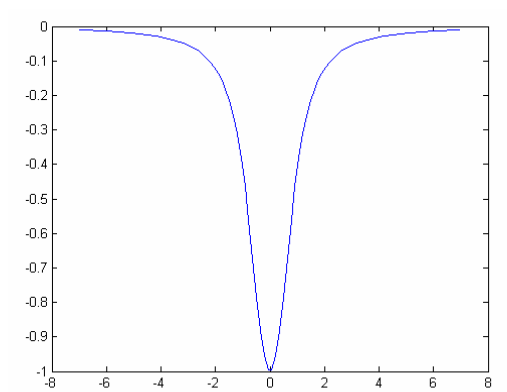
$$2xydx+(x^2-y)dy=0$$

<u>Observation</u>:

Step 1: Write $f(x,y)=0$ as: $y=x^2+\sqrt{x^4+1}$ and $y=x^2-\sqrt{x^4+1}$

Step 2: The graphs of these two functions are:





<u>Family of Solutions of an ODE</u>: Consider the ODE $\frac{dy}{dx}=1$. Then $y=x+c$ is a solution of the ODE for each value of constant c.

For each c, we get a different solution, so an ODE can have infinitely many solutions.

Note that: $y=x+1$, $y=x+2$ and $y=x+3$ etc are all solutions of the same ODE. In general, the function $f(x,y,c)=y-x-c=0$ gives ***one parameter family of solutions*** of the given ODE.

The solutions given by $y=x+1$, $y=x+2$ and $y=x+3$ are called ***particular solutions*** of the ODE.

Exercise: Verify that $y_1 = \cos 2x$, $y_2 = \sin 2x$ are both solutions of

$$y'' + 4y = 0.$$

Can you think of anymore solutions?

Question 22/10: Verify that the family of functions $y = e^{-x^2}\int_0^x e^{t^2} dt + c_1 e^{-x^2}$ is a solution of the ODE $\frac{dy}{dx} + 2xy = 1$.

Question 25/11: Verify that the piecewise-defined function

$$y = \begin{cases} -x^2, & x < 0 \\ x^2 \;, & x \geq 0 \end{cases}$$

is a solution of the DE $xy' - 2y = 0$ on $(-\infty, \infty)$.

Question 27(b)/10: Find the value of m so that $y = e^{mx}$ is a solution of the ODE $y'' - 5y' + 6y = 0$.

Question 29/11: Use the concept that $y = c$, $-\infty < x < \infty$ is a constant function if and only if $y' = 0$ to determine whether $3xy' + 5y = 10$ possesses a constant solution.

Systems of Differential Equations: A system of ODEs is two or more equations involving the derivatives of two or more unknown functions of a single independent variable.

Example: If x and y are dependent variable and t is independent variable, then a system of two first order differential equations is given by

$$\frac{dx}{dt} = f(t, x, y) \text{ and } \frac{dy}{dt} = g(t, x, y)$$

A ***solution*** of above system is a pair of differentiable functions $x = \phi_1(t)$, $y = \phi_2(t)$, defined on a common interval I, that satisfy each equation of the system on this interval.

Question 34/11: Verify that $x(t) = \cos 2t + \sin 2t + (e^t/5)$ and $y(t) = -\cos 2t - \sin 2t - (e^t/5)$ are solutions of the system $\frac{d^2x}{dt^2} = 4y + e^t$ and $\frac{d^2y}{dt^2} = 4x - e^t$.