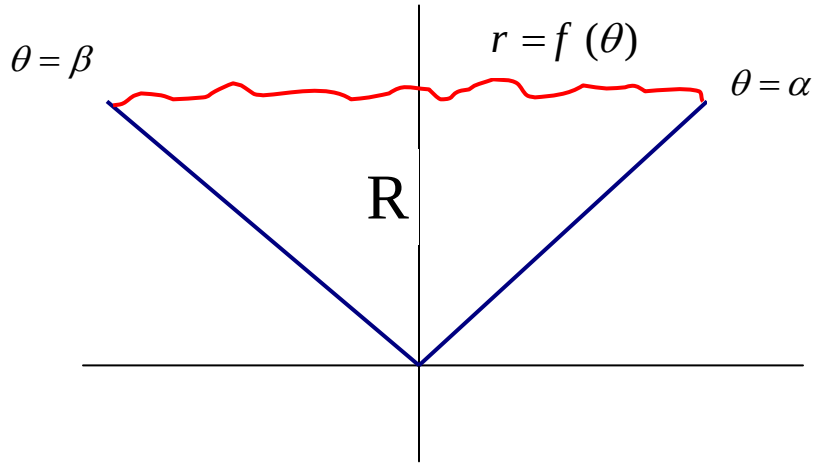
10.4 Areas and Lengths in Polar Coordinates



The area A of the region R enclosed by the polar curve $r = f(\theta)$ from the rays $\theta = \alpha$ to $\theta = \beta$ is

$$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta}$$

Assume that α and β satisfying $\alpha < \beta \le \alpha + 2\pi$, $f(\theta)$ is continuous and $f(\theta) \ne 0$ for $\alpha < \theta < \beta$

Procedure for finding the area:

- Sketch the curve
- Find α and β (See the explanation in class)
- Set up and evaluate integral

Caution About Finding Points of Intersections of Polar Curves

Equating two polar curves does not always give all points of intersection.

So the best strategy to find all points of intersections is

- to equate the equations and solve
- also to sketch the curve to see if there are more intersection points.

Why?

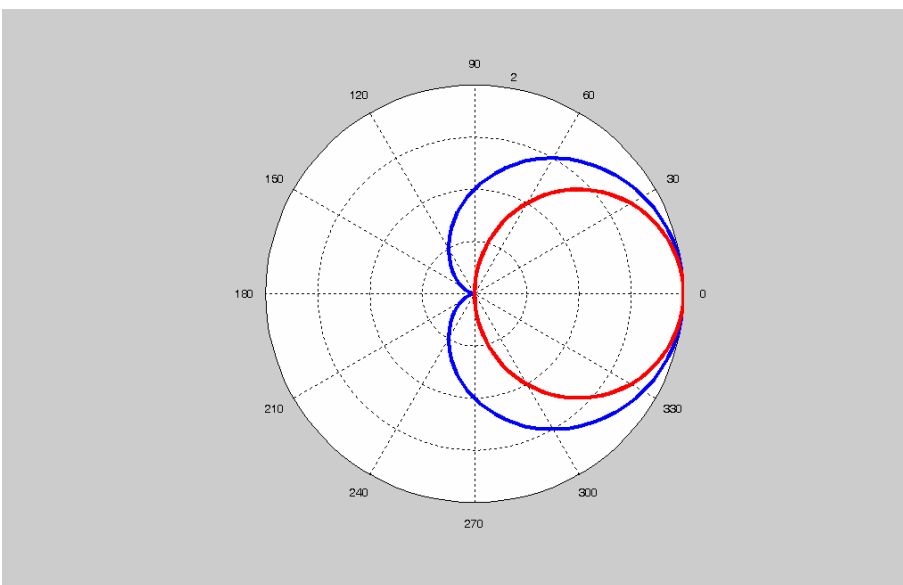
Because a point may have different representation as polar coordinates

Question: Find all points of intersections of $r = 1 + \cos\theta$ and $r = 2\cos\theta$ for

$$0 \le \theta < 2\pi .$$

Solution:

- By equating $1 + \cos\theta = \cos 2\theta$ we get $\cos\theta = 1 \quad \Rightarrow \quad \theta = 0$.
- From the sketch below, we see that "pole" is also a point of intersection.



Question 4/683: Find the area of the region that is bounded by the curve $r = \sqrt{\sin\theta}$ in the sector $0 \le \theta \le \pi$.

Question 19/683: Find the area of the region enclosed by the one loop of the curve $r = 3\cos 5\theta$.

Question 28/683: Find the area of the region that lies inside the curve $r = 1 + \cos\theta$ and outside the curve $r = 3\cos\theta$.

Question 32/683: Find the area of the region that lies inside both curves $r = 1$ and $r^2 = 2\sin 2\theta$.

Question 42/683: Find all points of intersection of the curves $r^2 = \sin 2\theta$ and $r^2 = \cos 2\theta$.

Arc Length Formula for Polar Curve

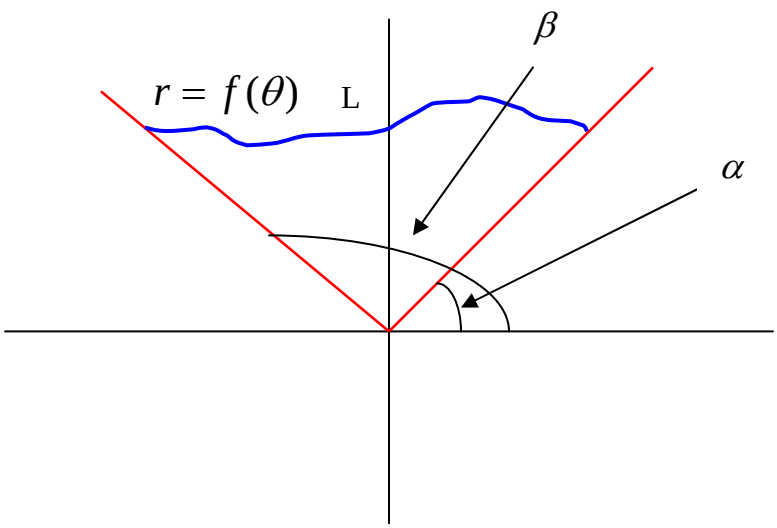
Given a polar curve $r = f(\theta)$

(θ increases from α to θ)

The arc length of the curve from $\theta = \alpha$ to $\theta = \beta$ is

$$\boxed{L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

We assume that curve is traced only once and $\alpha \le \theta \le \beta$



Question 46/684: Find the exact length of the polar curve $r = e^{2\theta}$ from $\theta = 0$ to $\theta = 2\pi$.