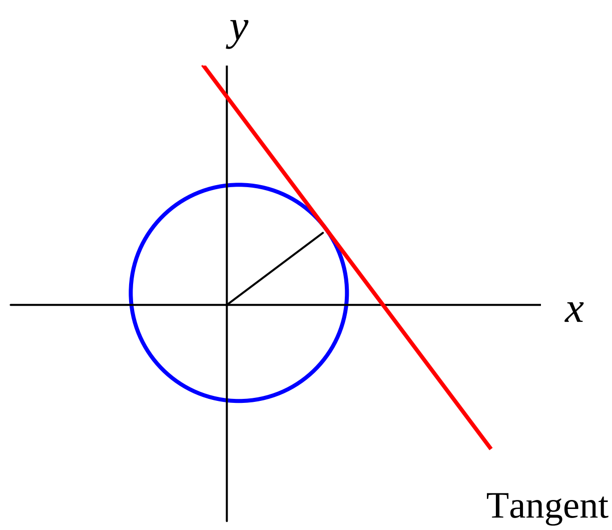
# 10.2 Calculus with Parametric Curves

## Tangent Lines to Parametric Curves




Consider the parametric equations of a curve

$$x = f(t), \quad y = g(t); \qquad a \le t \le b$$

- $\dfrac{dy}{dx}$ is defined as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{provided } \frac{dx}{dt} \neq 0$$

- Similarly

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

* slope of tangent line at $x = f(t_0),\ y = g(t_0)$

$$\left.\frac{dy}{dx}\right|_{t=t_0}$$

* Tangent is horizontal when

$$\frac{dy}{dt} = 0 \text{ and } \frac{dx}{dt} \neq 0$$

* Tangent is vertical when

$$\frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} \neq 0$$

* Points where $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$ are called singular points

* The curve is concave upward when $\frac{d^2y}{dx^2} \geq 0$ and concave downward when $\frac{d^2y}{dx^2} \leq 0$.

**Question 6/666:** Find an equation of the tangent to the curve $x = \cos\theta + \sin 2\theta,\ y = \sin\theta + \cos 2\theta$ at $\theta = 0$.

**Question 7/666:** Find an equation of the tangent to the curve $x = e^t,\ y = (t-1)^2$ at $(1,1)$ (a) without eliminating the parameter.

(b) by first eliminating the parameter.

**Question 15/666:** If $x = 2\sin t,\ y = 3\cos t,\ \ 0 < t < 2\pi$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of $t$ is the curve concave upward?

**Question 20/666:** Find the points on the curve $x = \cos 3\theta,\ y = 2\sin\theta$ where the tangent is horizontal or vertical.

**Question 29/667:** At what points on the curve $x = t^3 + 4t,\ y = 6t^2$ is the tangent parallel to the line with equation $x = -7t,\ y = 12t - 5$?

## Area Formula for Parametric Curves

The area A of the region R enclosed by the parametric curve

$x = f(t),\ \ y = g(t)$ as $t$ increases from $\alpha$ to $\beta$ is

$$A = \int_{\alpha}^{\beta} g(t) f'(t)\,dt$$

$$\left[ \text{or } A = \int_{\beta}^{\alpha} g(t) f'(t)\,dt \quad \text{if } (f(\beta), g(\beta)) \text{ is the leftmost endpoint} \right]$$

**Question 34/667:** Find the area of the region enclosed by the asteroid $x = a\cos^3\theta,\ \ y = a\sin^3\theta$.

## Arc Length Formula for Parametric Curves

Given a parametric curve $x = f(t)$, $y = g(t)$, as $t$ increases from $\alpha$ to $\beta$ or $(\alpha \le t \le \beta)$.

The arc length of the curve from $t = \alpha$ to $t = \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

**Question 42/667:** Find the length of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, $(0 \le \theta \le \pi)$.

## Surface area Generated by Revolving the Curve

Given parametric equations of a curve

$$x = f(t), \quad y = g(t) \qquad (\alpha \le t \le \beta)$$

then the area of the surface generated by revolving this curve about $X$ -axis is

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

and the area of the surface generated by revolving this curve about $Y$ -axis is

$$S = \int_{\alpha}^{\beta} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

We assume that curve is traced only once and $f'(t)$ and $g'(t)$ are continuous

**Question 61/668:** Find the area of the surface generated by revolving the curve

$$x = a\cos^3\theta, \quad y = a\sin^3\theta \qquad (0 \le \theta \le \pi/2)$$

about $X$ -axis.