15.8 Triple Integrals in Cylindrical and Spherical Coordinates

We divide this section into two parts. First part deals with the study of triple integrals in cylindrical coordinates while second part presents the study of triple integrals in spherical coordinates.

(Part 1)

Triple Integrals in Cylindrical Coordinates

Recall

- Cylindrical coordinates: (r, θ, z)
- Conversion formulas for cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Aim: Learn to integrate $f(r, \theta, z)$

- Recall area element in polar coordinates: $dA = r\,dr\,d\theta$
- Volume element in cylindrical coordinates $dV = r\,dz\,dr\,d\theta$

Evaluating Triple Integrals in Cylindrical Coordinates

Given a function $f(r,\theta,z)$ over a solid G such that

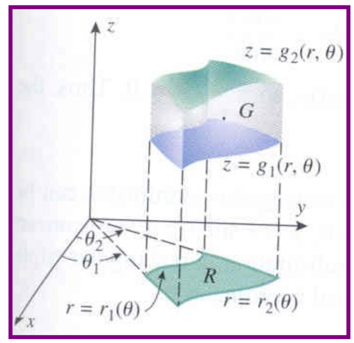
- G is bounded above by $z = g_2(r,\theta)$ and below by $z = g_1(r,\theta)$
- and the projection R of G on XY-plane is a simple polar region.

Then the triple integral is evaluated as follows:

$$\iiint_G f(r,\theta,z)\,dV = \iint_R \left[\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\,dz\right] dA$$

or

$$\iiint_G f(r,\theta,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\,r\,dz\,dr\,d\theta$$



Find Limit of Integrals

Step 1. Identify the upper surface $z = g_2(r,\theta)$ and the lower surface $z = g_1(r,\theta)$ of the solid. The functions $g_1(r,\theta)$ and $g_2(r,\theta)$ determine the z-limits of integration. (If the upper and lower surfaces are given in rectangular coordinates, convert them to cylindrical coordinates.)

Step 2. Make a two dimensional sketch of the projection R of the solid on the xy-plane. From this sketch the $r-$ and $\theta-$limits of integration may be obtained exactly as with double integrals in polar coordinates.

Question 2/1037: Evaluate $TI = \int_0^{\pi/2}\int_0^2\int_0^{9-r^2} r\,dz\,dr\,d\theta$

Converting Triple Integrals from Rectangular to Cylindrical Coordinates

Some triple integrals are easier to evaluate in cylindrical coordinates.

Especially,

- If the expressions of the form $x^2 + y^2 = a^2$ appear in the limits or integrand

How to convert?

- Use $\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix} \Rightarrow x^2 + y^2 = r^2$
- and convert limits

Question 34/1038: Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$ by converting to cylindrical coordinates.

Question 11/1038: Evaluate $\iiint_G x^2 dV$, where G is the solid that lies within the Cylinder $x^2 + y^2 = 1$, above the plane $z = 0$ and below the cone $z^2 = 4x^2 + 4y^2$ by converting to cylindrical coordinates.

Solution: Done in class

Important steps for conversion

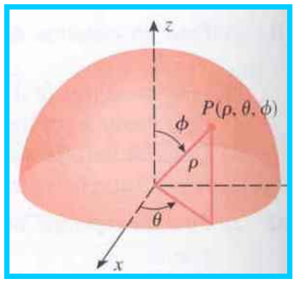
- identify the regions G, R
- then find limits of integration in cylindrical coordinates

(Part 2)

Triple Integrals in Spherical Coordinates

Recall

- Cylindrical coordinates: (ρ,θ,ϕ)
- $\rho = \text{constant}$: sphere
- $\phi = \text{constant}$: cone



Aim: Learn to integrate $f(\rho,\theta,\phi)$

Volume element in cylindrical coordinates $dV = \rho^2 \sin\phi d\rho d\phi d\theta$

Evaluating Triple Integrals in Spherical Coordinates

Evaluated as triple integral

$$\iiint\limits_G f(\rho,\theta,\phi)dV = \iiint\limits_{\text{limits}} f(\rho,\theta,\phi)\rho^2 \sin\phi d\rho d\phi d\theta$$

We learn the techniques of putting limits through examples

Question 3/1036: Evaluate $TI = \int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho \sin\phi d\rho d\theta d\phi$

Converting Triple Integrals from Rectangular to Spherical Coordinates

Some triple integrals are easier to evaluate in spherical coordinates.

Especially,

- If the expressions of the form $x^2 + y^2 + z^2 = a^2$ appear in the limits or integrand

How to convert?

- Use $$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned} \quad \Rightarrow \quad x^2 + y^2 + z^2 = \rho^2$$
- and convert limits

Question 36/1038: Evaluate $\displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2)\,dz\,dx\,dy$

by converting to spherical coordinates.

Question 19/1038: Evaluate $TI = \iiint_G z dV$ where G lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant by converting to spherical coordinates.

Solution: Done in class

Important steps for conversion

- identify the regions G, R
- then find limits of integration in cylindrical coordinates

Question 24/1038: Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the XY-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

End of the course

You have the license to steer all models of Math201