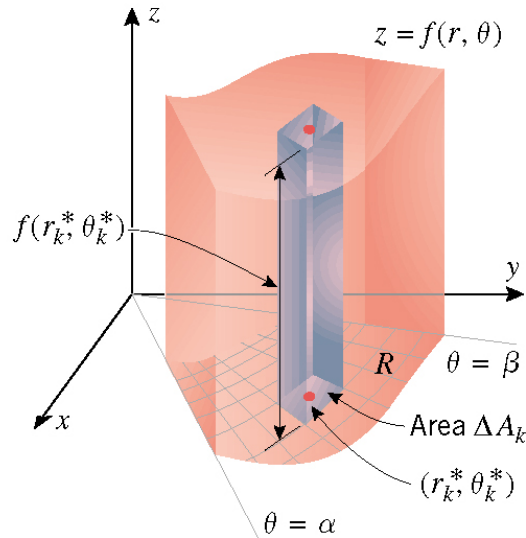


## 15.4 Double Integrals in Polar Coordinates

### Evaluating Double Integrals in Polar Coordinates

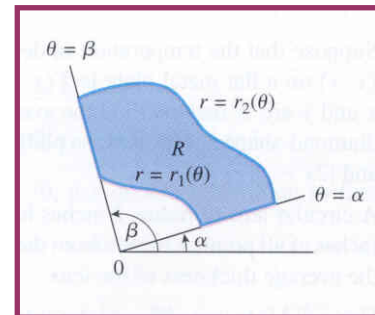


#### Simple polar region

A region defined by

- $\{(r, \theta) : \alpha \leq \theta \leq \beta \text{ and } 0 \leq r_1(\theta) \leq r \leq r_2(\theta)\}$

where  $\beta - \alpha \leq 2\pi$



#### Formula for evaluating double integral in polar coordinates

Given  $f(r, \theta)$  over a simple region  $R$  given by

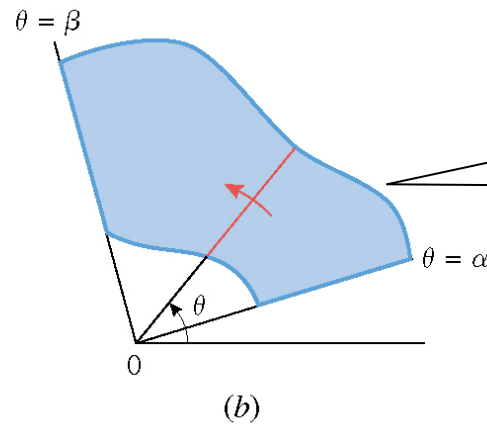
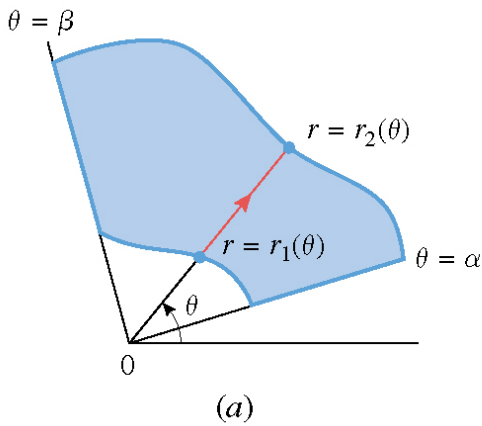
$$\{(r, \theta) : \alpha \leq \theta \leq \beta \text{ and } 0 \leq r_1(\theta) \leq r \leq r_2(\theta)\}.$$

Then

$$\iint_R f(r, \theta) dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) r dr d\theta$$

**Note: Area element  $dA$**

- in rectangular coordinates:  $dA = dx dy$
- in polar coordinates:  $dA = r dr d\theta$



**Find limit of Integrals for polar region R**

**Step 1.** Draw a radial line from the origin through the region  $R$  at a fixed angle  $\theta$ . This line intersects the region on the inner boundary curve  $r = r_1(\theta)$  and outer boundary curve  $r = r_2(\theta)$ . Then  $r = r_1(\theta)$  is the lower limit and  $r = r_2(\theta)$  is the upper limit of the inner integral.

**Step 2.** Rotate a ray along the polar X-axis one revolution counter-clockwise about the origin. The smallest angle at which this ray intersects the region  $R$  is  $\theta = \alpha$  and the largest angle is  $\theta = \beta$ .  $\theta = \alpha$  is the lower limit and  $\theta = \beta$  is the upper limit of the outer integral.

**Question 7/1008:** Evaluate  $\int_{\pi}^{2\pi} \int_4^7 r dr d\theta$ .

## Application

- If  $f(r, \theta) \geq 0$  then  $\iint_R f(r, \theta) dA$  gives volume under surface  $z = f(r, \theta)$  over the region  $R$ .
- $\iint_R dA$  gives area of the region  $R$ .

**Question 19/1008:** Use double integral to find the area of the region within both of the circles  $r = \cos \theta$  and  $r = \sin \theta$ .

## Converting Double Integrals into Polar Coordinates

Some double integrals are easier to evaluate in polar coordinates.

Especially,

- If the region  $R$  consists of a circle or a part of the circle.  
i.e contains an equation of the form  $x^2 + y^2 = a^2$
- If the integrand contains expressions of the form  $x^2 + y^2 = a^2$ .

### How to convert?

Use

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \Rightarrow x^2 + y^2 = r^2$$

**Question 15/1008:** Use polar coordinates to evaluate  $\iint_R \arctan(y/x) dA$ , where

$$R = \{(x, y) : 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}.$$

**Question 25/1008:** Find the volume of the solid above the cone

$z = \sqrt{x^2 + y^2}$  and below the sphere  $x^2 + y^2 + z^2 = 1$  by converting to polar coordinates.

**Question 31/1009:** Evaluate the iterated integral  $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2 y^2 dx dy$  by

converting to polar coordinates.

**Exercise**

Find  $I = \iint_R e^{x^2+y^2} dA$  over the circle  $x^2 + y^2 = 1$

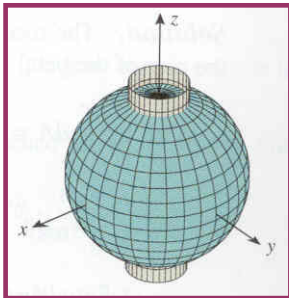
Answer:  $I = \int_0^{2\pi} \int_0^1 r e^{r^2} dr d\theta = \pi(e-1)$

**Question:**

Find the volume of solid that lies under the sphere  $x^2 + y^2 + z^2 = 9$ , above the plane  $z = 0$  and inside the cylinder  $x^2 + y^2 = 4$

**Hint:**

- Figure:



- In rectangular coordinates the volume is given by  $V = \iint_R \sqrt{9 - x^2 - y^2} dA$  where

$R$  is the region inside the circle  $x^2 + y^2 = 4$ .

Not a nice integration

- Do we get easier integration in polar coordinates?