14.6 Directional Derivatives and the Gradient Vector

We have done partial derivatives
- \( f_x \): rate of change of \( f \) in x-direction
- \( f_y \): rate of change of \( f \) in y-direction

In this section, we will see the directional derivatives
- rate of change of \( f \) in any given direction

Before defining the directional derivative, we study the gradient of a function of two or more variables

Gradient of a function

Also called gradient of \( f \)

For a function \( f(x, y, z) \), the gradient vector of \( f(x, y, z) \) is defined as

\[
\nabla f = \langle f_x, f_y, f_z \rangle
\]

\[
\nabla f \ (x, y, z) = f_x (x, y, z)i + f_y (x, y, z)j + f_z (x, y, z)k
\]

Similar definition can be defined for functions of two or more variables.

\( \nabla f \) is a vector
Properties of $\nabla f$: (Same as derivatives)

If $f$ and $g$ are differentiable, then

- $\nabla (f \pm g) = \nabla f \pm \nabla g$
- $\nabla (cf) = c \nabla f$ (where $c$ is any constant)
- $\nabla (fg) = f \nabla g + g \nabla f$
- $\nabla \left( \frac{f}{g} \right) = \frac{g \nabla f - f \nabla g}{g^2}$
- $\nabla (f^n) = n f^{n-1} \nabla f$

Important Fact About Gradient

Given a surface $z = f(x, y)$.

- Of course the direction $\nabla f(x_0, y_0)$ is important
- How to determine this direction from the contour map of $f(x, y)$

Recall that for $z = f(x, y)$ we can find the level curve $f(x, y) = k$ that passes through the point

- Let $z = f(x, y)$ be a surface and $f(x, y) = k$
  be the level curve that passes through $(x_0, y_0)$.
- Then $\nabla f(x_0, y_0)$ is normal to the level curve $f(x, y) = k$. 
Let $f(x, y, z)$ be differentiable at $(x_0, y_0, z_0)$ and $\nabla f(x_0, y_0, z_0) \neq \vec{0}$.

Then $\nabla f(x_0, y_0, z_0)$ is normal to the level surface of $f$ through $(x_0, y_0, z_0)$.

**Question 9/951**: Find the gradient of $f(x, y, z) = xe^{2yz}$ at $P(3, 0, 2)$.

Also find the rate of change of $f(x, y, z)$ at $P$ in the direction of the vector $\vec{u} = (2/3, -2/3, 1/3)$. 
What is Directional Derivative in the Direction of a Vector $\mathbf{u}$ at $(x_0, y_0)$?

- Slope of surface $z = f(x, y)$ at $(x_0, y_0)$ in the direction of $\mathbf{u}$
- Rate of change of $z = f(x, y)$ at $(x_0, y_0)$ in the direction of $\mathbf{u}$

Main question:
How to compute directional derivatives?

The directional derivative of $f(x, y)$ at $(x_0, y_0)$ in the direction of a unit vector $\mathbf{u} = <u_1, u_2>$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

provided that this limit exists.
How to Compute Directional Derivative

The directional derivative of $f(x,y)$ in the direction of unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is given by

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)u_1 + f_y(x,y)u_2$$

Assumption

$f(x,y)$ differentiable

The directional derivative of $f(x,y,z)$ in the direction of unit vector $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is given by

$$D_{\mathbf{u}} f(x,y,z) = f_x(x,y,z)u_1 + f_y(x,y,z)u_2 + f_z(x,y,z)u_3$$

Similar formulas for more variables

Directional Derivative in terms of Gradient

The directional derivative of $f(x,y,z)$ in the direction of unit vector $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is given by

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$
Question 5/950: Find the direction derivative of \( f(x, y) = \sqrt{5x - 4y} \) at \( P(4,1) \) in the direction indicated by the angle \( \theta = -\pi/6 \).

Question 16/951: Find the direction derivative of \( f(x, y, z) = \frac{x}{y + z} \) at \( P(4,1,1) \) in the direction of the vector \( \vec{v} = (1,2,3) \).
Important Fact

Gradient vector determines the maximum / minimum rate of change of a function

Let \( f \) be a function of 2 or 3 variables.

- The maximum value (minimum value) of the directional derivative of \( f \) occurs in direction (opposite to that) of gradient vector \( \nabla f \).
- Hence, the maximum value (minimum value) of the directional derivative of \( f \) (i.e. maximum (minimum) rate of change of \( f \)) is \( -|\nabla f| \) (**-|\nabla f|**).

Why?
Since \( D_u f = \nabla f \cdot u = |\nabla f| |u| \cos \theta = |\nabla f| \cos \theta \)

**Question 26/951:** Find the maximum rate of change of \( f (x, y, z) = \tan(x + 2y + 3z) \) at the point \( P(-5,1,1) \) and the direction in which it occurs.
What are tangent planes & normal lines?

Equation of Tangent Plane & Normal Line to a Surface

- Given a surface $z = f(x, y)$, we can write it as $F(x, y, z) = c$.
  - i.e. we can think of the surface $z = f(x, y)$ as a level surface $F(x, y, z) = c$ of a function of three variables.
  - Hence, $\nabla F(x_0, y_0, z_0)$ will be normal to the surface $F(x, y, z) = c$ or $z = f(x, y)$ at $(x_0, y_0, z_0)$.

Given the surface $F(x, y, z) = c$.

- The vector $\nabla F(x_0, y_0, z_0)$ is normal to surface at $(x_0, y_0, z_0)$.
- The equation of the tangent plane at $(x_0, y_0, z_0)$ is
  \[ F_x(x_0, y_0, z_0)(x-x_0) + F_y(x_0, y_0, z_0)(y-y_0) + F_z(x_0, y_0, z_0)(z-z_0) = 0 \]
- The equation of the normal line at $(x_0, y_0, z_0)$ is
  \[ x = x_0 + F_x(x_0, y_0, z_0)t, \quad y = y_0 + F_y(x_0, y_0, z_0)t, \quad z = z_0 + F_z(x_0, y_0, z_0)t \]
The symmetric equation of the normal line is

\[
\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}
\]

**Question 44/952**: Find equation of (a) the tangent plane and (b) the normal line to the surface \(yz = \ln(x + z)\) at \(P(0,0,1)\).

**Question 53/952**: Find the points on the hyperboloid \(x^2 - y^2 + 2z^2 = 1\) where the normal line is parallel to the line that joins the points \(P(3,-1,0)\) and \(Q(5,3,6)\).