14.6 Directional Derivatives and the Gradient Vector

We have done partial derivatives

- f_x : rate of change of 'f' in x-direction
- f_y : rate of change of 'f' in y-direction

In this section, we will see the directional derivatives

- rate of change of 'f' in any given direction

Before defining the directional derivative, we study the gradient of a function of two or more variables

Gradient of a function

Also called gradient of 'f'

For a function $f(x, y, z)$, the gradient vector of $f(x, y, z)$ is defined as

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$\nabla f(x, y, z) = f_x(x, y, z)\vec{i} + f_y(x, y, z)\vec{j} + f(x, y, z)\vec{k}$$

Similar definition can be defined for functions of two or more variables.

∇f is a vector

Properties of ∇f : (Same as derivatives)

If f and g are differentiable, then

- $\nabla(f \pm g) = \nabla f \pm \nabla g$
- $\nabla(cf) = c\nabla f$ (where c is any constant)
- $\nabla(fg) = f\nabla g + g\nabla f$
- $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$
- $\nabla(f^n) = nf^{n-1}\nabla f$

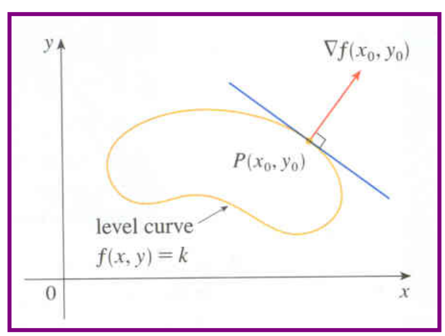
Important Fact About Gradient

Given a surface $z = f(x,y)$.

- Of course the direction $\nabla f\,(x_0, y_0)$ is important
- How to determine this direction from the contour map of $f(x,y)$

Recall that for $z = f(x,y)$ we can find the level curve $f(x,y) = k$ that passes through the point

- Let $z = f(x,y)$ be a surface and $f(x,y) = k$ be the level curve that passes through (x_0, y_0).
- Then $\nabla f\,(x_0, y_0)$ is normal to the level curve $f(x,y) = k$.



- Let $f(x,y,z)$ be differentiable at (x_0,y_0,z_0) and $\nabla(x_0,y_0,z_0)\neq\vec{0}$.
- Then $\nabla f\left(x_0,y_0,z_0\right)$ is normal to the level surface of f through (x_0,y_0,z_0).

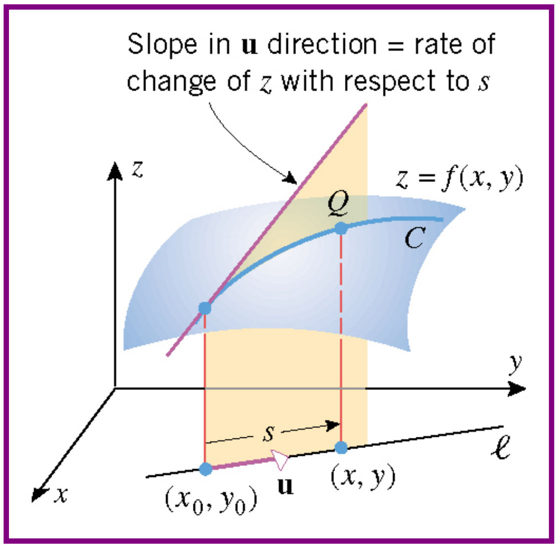
Question 9/951: Find the gradient of $f(x,y,z)=xe^{2yz}$ at $P(3,0,2)$.

Also find the rate of change of $f(x,y,z)$ at P in the direction of the vector $\vec{u}=\langle 2/3,-2/3,1/3\rangle$.

What is Directional Derivative

in the Direction of a Vector $\vec{\mathbf{u}}$ at (x_0, y_0)?

- Slope of surface $z = f(x,y)$ at (x_0, y_0) in the direction of $\vec{\mathbf{u}}$
- Rate of change of $z = f(x,y)$ at (x_0, y_0) in the direction of $\vec{\mathbf{u}}$



Main question:

How to compute directional derivatives?

The directional derivative of $f(x, y)$ at (x_0, y_0) in the direction of a **unit vector** $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

provided that this limit exists

How to Compute Directional Derivative

The directional derivative of $f(x,y)$ in the direction of **unit vector** $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ is given by

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)u_1 + f_y(x,y)u_2$$

Assumption

$f(x,y)$ differentiable

The directional derivative of $f(x,y,z)$ in the direction of **unit vector** $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ is given by

$$D_{\mathbf{u}} f(x,y,z) = f_x(x,y,z)u_1 + f_y(x,y,z)u_2 + f_z(x,y,z)u_3$$

Similar formulas for more variables

Directional Derivative in terms of Gradient

The directional derivative of $f(x,y,z)$ in the direction of **unit vector** $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ is given by

$$D_{\mathbf{u}} f = \nabla f \cdot \vec{\mathbf{u}}$$

Question 5/950: Find the direction derivative of $f(x,y)=\sqrt{5x-4y}$ at $P(4,1)$ in the direction indicated by the angle $\theta=-\pi/6$.

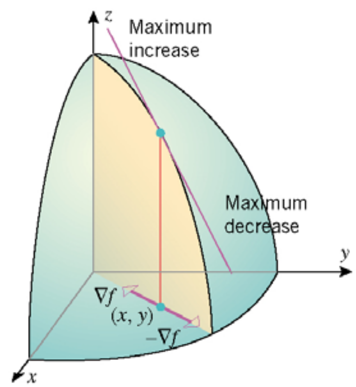
Question 16/951: Find the direction derivative of $f(x,y,z)=\dfrac{x}{y+z}$ at $P(4,1,1)$ in the direction of the vector $\vec{v}=\langle 1,2,3\rangle$.

Important Fact

Gradient vector determines the maximum / minimum rate of change of a function

Let 'f' be a function of 2 or 3 variables.

- The maximum value (minimum value) of the directional derivative of 'f' occurs in direction (opposite to that) of gradient vector ∇f .
- Hence, the maximum value (minimum value) of the directional derivative of 'f' (i.e. maximum (minimum) rate of change of 'f') is '$|\nabla f|$' ('$-|\nabla f|$').



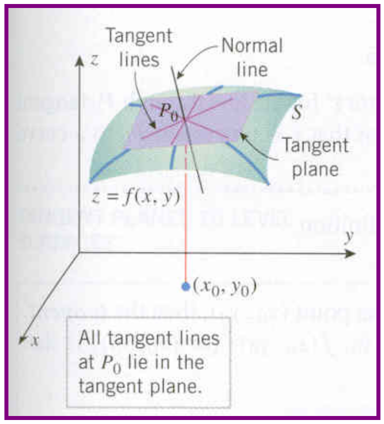
Why?

Since $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$

Question 26/951: Find the maximum rate of change of $f(x,y,z) = \tan(x+2y+3z)$ at the point $P(-5,1,1)$ and the direction in which it occurs.

Tangent Planes to Level Surfaces

What are tangent planes & normal lines?



Equation of Tangent Plane & Normal Line to a Surface

- Given a surface $z = f(x,y)$, we can write it as $F(x,y,z) = c$.
  - i.e. we can think of the surface $z = f(x,y)$ as a level surface $F(x,y,z) = c$ of a function of three variables.
  - Hence, $\nabla F(x_0, y_0, z_0)$ will be normal to the surface $F(x,y,z) = c$ or $z = f(x,y)$ at (x_0, y_0, z_0).

Given the surface $F(x,y,z) = c$.

- The vector $\nabla F(x_0, y_0, z_0)$ is normal to surface at (x_0, y_0, z_0).
- The equation of the tangent plane at (x_0, y_0, z_0) is

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

- The equation of the normal line at (x_0, y_0, z_0) is

$$x = x_0 + F_x(x_0,y_0,z_0)t, \quad y = y_0 + F_y(x_0,y_0,z_0)t, \quad z = z_0 + F_z(x_0,y_0,z_0)t$$

What do we need for equation of plane and equation of line ?

The symmetric equation of the normal line is

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Question 44/952: Find equation of (a) the tangent plane and (b) the normal line to the surface $yz = \ln(x + z)$ at $P(0,0,1)$.

Question 53/952: Find the points on the hyperboloid $x^2 - y^2 + 2z^2 = 1$ where the normal line is parallel to the line that joins the points $P(3,-1,0)$ and $Q(5,3,6)$.