14.5 The Chain Rules

Recall chain rule for functions of one variable

If $y = f(x)$ and $x = g(t)$ then $\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$

- Here we study chain rule for functions of more variables
- It has different versions

Chain Rule Case-I

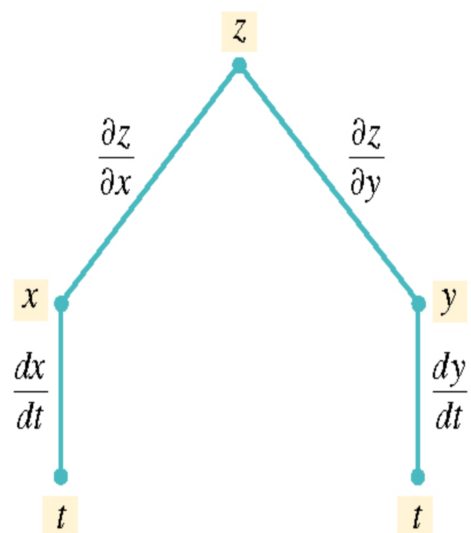
If $z = f(x, y)$ and $x = g(t),\ y = h(t)$ then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

If $w = f(x, y, z)$ and $x = g(t),\ y = h(t)$

$z = k(t)$ then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

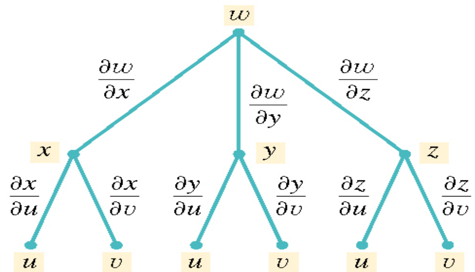
Question 5/938: If $w = xe^{y/z}$ and $x = t^2$, $y = 1 - t$, $z = 1 + 2t$, then find dw/dt by using an appropriate form of the chain rule.

Chain Rule Case-II

If $z = f(x,y)$ and $x = g(u,v)$, $y = h(u,v)$

then $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

If $w = f(x,y,z)$ and $x = g(u,v)$, $y = h(u,v)$, $z = k(u,v)$ then

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

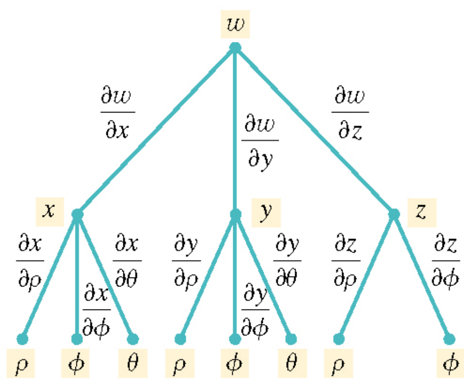
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

Question 12/938: Use Chain rule to find $\partial z/\partial s$ and $\partial z/\partial t$ if $z = \sin\alpha \tan\beta$ and $\alpha = 3s + t$, $\beta = s - t$.

Similar to above logic you should be able to develop the chain rule for different situations.

If $w = f(x, y, z)$ is a function of $x = x(\rho,\phi,\theta), y = (\rho,\phi,\theta)$ and $z = (\rho,\phi)$, then the relevant formula is

$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho} \quad \text{and} \quad \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}$$



$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}$$

Exercise Let $w = f(x,y,z)$ and $x = g(t,u,v),\ y = h(t,u,v),\ z = k(t,u,v)$.

Write down the chain rule formula for $\frac{\partial w}{\partial t}$.

Answer: $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}$

If u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

Question 15/938: If $u = x^2 + yz$, $x = pr\cos\theta$, $y = pr\sin\theta$, and $z = p + r$. Then find $\partial u/\partial p$, $\partial u/\partial r$, $\partial u/\partial\theta$ by using appropriate Chain rule.

Implicit Differentiation

- Given $F(x,y)=C$

Defining y implicitly as function of x

- Differentiating w.r.t. x we get

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-F_x}{F_y}$$

If $F(x,y)=C$ implicitly defines y as a function of x then

$$\frac{dy}{dx}=\frac{-F_x}{F_y} \qquad (\text{if } F_y\neq 0)$$

Question 30/938: If $\sin x+\cos y=\sin x\cos y$. Find $\dfrac{dy}{dx}$.

Defining z implicitly as function of x and y

- Consider $F(x,y,z)=C$ (*)
  - Differentiate (using chain rule) Eq. (*) w.r.t. x, we have

$$\frac{\partial z}{\partial x}=\frac{-F_x}{F_z}$$

  - Differentiate (using chain rule) Eq. (*) w.r.t. y, we have

$$\frac{\partial z}{\partial y}=\frac{-F_y}{F_z}$$

Question 33/938: If $x-z=\arctan(yz)$. Find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$.