14.5 The Chain Rules

Recall chain rule for functions of one variable

If $y = f(x)$ and $x = g(t)$ then \[ \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \]

- Here we study chain rule for functions of more variables
- It has different versions

Chain Rule Case-I

If $z = f(x, y)$ and $x = g(t), y = h(t)$ then

\[ \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \]

If $w = f(x, y, z)$ and $x = g(t), y = h(t)$

$z = k(t)$ then

\[ \frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt} \]

**Question 5/938:** If $w = xe^{y/z}$ and $x = t^2$, $y = 1 - t$, $z = 1 + 2t$, then find $dw/dt$ by using an appropriate form of the chain rule.
If \( z = f(x,y) \) and \( x = g(u,v), \ y = h(u,v) \), then
\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}
\]
\[
\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}
\]

If \( w = f(x,y,z) \) and \( x = g(u,v), \ y = h(u,v), \ z = k(u,v) \) then
\[
\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}
\]
\[
\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}
\]

**Question 12/938:** Use Chain rule to find \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \) if \( z = \sin \alpha \tan \beta \) and \( \alpha = 3s + t, \ \beta = s - t \).
If \( w = f(x, y, z) \) is a function of \( x = x(\rho, \phi, \theta), y = (\rho, \phi, \theta) \) and \( z = (\rho, \phi) \), then the relevant formula is

\[
\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial \rho}
\]
and

\[
\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial \theta}
\]

Similar to above logic you should be able to develop the chain rule for different situations.

**Exercise**

Let \( w = f(x, y, z) \) and \( x = g(t, u, v), y = h(t, u, v), z = k(t, u, v) \).

Write down the chain rule formula for \( \frac{\partial w}{\partial t} \).

**Answer:**

\[
\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t}
\]
If $u$ is a differentiable function of the $n$ variables $x_1, x_2, \ldots, x_n$ and each $x_j$ is a differentiable function of the $m$ variables $t_1, t_2, \ldots, t_m$. Then $u$ is a function of $t_1, t_2, \ldots, t_m$ and

$$
\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}
$$

for each $i = 1, 2, \ldots, m$.

**Question 15/938:** If $u = x^2 + yz$, $x = pr \cos \theta$, $y = pr \sin \theta$, and $z = p + r$. Then find $\frac{\partial u}{\partial p}$, $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial \theta}$ by using appropriate Chain rule.
Implicit Differentiation

- Given \( F(x, y) = C \)

- Differentiating w.r.t. \( x \) we get

\[
\frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0
\]

\[
\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}
\]

\[\text{Defining } y \text{ implicitly as function of } x\]

- If \( F(x, y) = C \) implicitly defines \( y \) as a function of \( x \) then

\[
\frac{dy}{dx} = \frac{-F_x}{F_y} \quad \text{(if } F_y \neq 0)\]

\[\text{Question 30/938: If } \sin x + \cos y = \sin x \cos y \text{. Find } \frac{dy}{dx}.\]

- Consider \( F(x, y, z) = C \) (*)

- Differentiate (using chain rule) Eq. (*) w.r.t. \( x \), we have

\[
\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}
\]

- Differentiate (using chain rule) Eq. (*) w.r.t. \( y \), we have

\[
\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}
\]

\[\text{Defining } z \text{ implicitly as function of } x \text{ and } y\]

- Question 33/938: If \( x - z = \arctan(yz) \). Find \( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \).