14.3 Partial derivatives

Formal Definition

- The partial derivative of $f(x,y)$ w.r.t. 'x' at the point (x_0, y_0) is denoted as $\frac{\partial f}{\partial x}$ and defined as

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

Rate of change in 'x' direction

- The partial derivative of $f(x,y)$ w.r.t. 'y' at the point (x_0, y_0) is denoted as $\frac{\partial f}{\partial y}$ and defined as

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Rate of change in 'y' direction

Informal Definition $\frac{\partial f}{\partial x}$

Ordinary derivative of $f(x,y)$ w.r.t. 'x'

- keeping 'y' as constant

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Ordinary derivative of $f(x,y)$ w.r.t. 'y'

- keeping 'x' as constant

Notation

Consider $z = f(x, y)$.

- The partial derivatives w.r.t. 'x' and 'y' are denoted respectively by

$$f_x, \ \frac{\partial f}{\partial x}, \ \frac{\partial z}{\partial x} \qquad \text{and} \qquad f_y, \ \frac{\partial f}{\partial y}, \ \frac{\partial z}{\partial y}$$

- The partial derivatives at the point (x_0, y_0) are

$$f_x(x_0, y_0), \ \left.\frac{\partial f}{\partial x}\right|_{x=x_0,\ y=y_0}, \ \left.\frac{\partial z}{\partial x}\right|_{(x_0,\ y_0)} \qquad \text{and} \qquad f_y(x_0, y_0), \ \left.\frac{\partial f}{\partial y}\right|_{x=x_0,\ y=y_0}, \ \left.\frac{\partial z}{\partial y}\right|_{(x_0,\ y_0)}$$

Question 17/921: Let $f(x, y) = \dfrac{x - y}{x + y}$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$

Question 31/921: Find the partial derivatives of the function

$f(x, y, z, t) = xyz^2 \tan(yt)$.

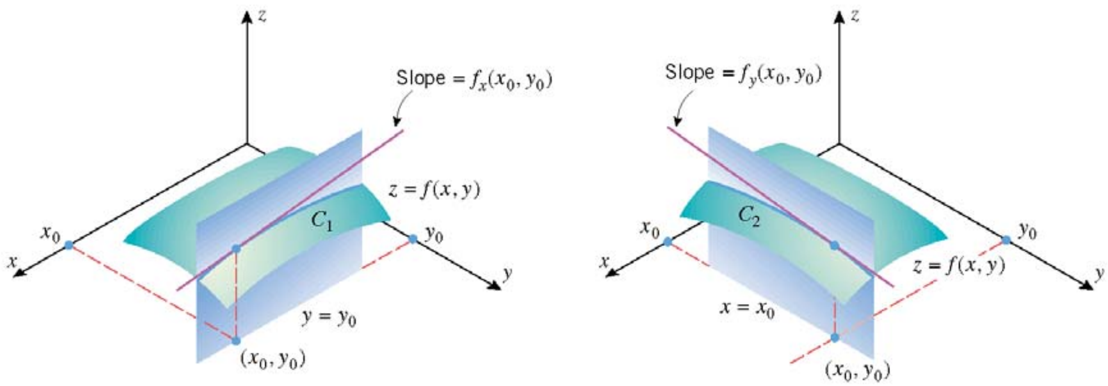
Question 33/921: Consider $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Find $\dfrac{\partial u}{\partial x_i}$.

Implicit partial differentiation

- Learn through an example

Question 44/921: If $\sin(xyz) = x + 2y + 3z$ then calculate $\partial z / \partial x$ and $\partial z / \partial y$.

Interpretation of partial derivatives of z = f(x,y)



$\frac{\partial f}{\partial x}(x_0, y_0)$

- **rate of change** of $f(x,y)$ in **x-direction**
- **slope** of surface $z=f(x,y)$ in **x-direction**

$\frac{\partial f}{\partial y}(x_0, y_0)$

- **rate of change** of $f(x,y)$ in **y-direction**
- **slope** of surface $z=f(x,y)$ in **y-direction**

Question 9/920: If $f(x,y) = 16 - 4x^2 - y^2$, find $f_x(1,2)$ and $f_y(1,2)$ and interpret these numbers as slope.

Higher order partial derivatives

- Since $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are also functions of x & y, so we can differentiate them further
- For $z=f(x,y)$, the four second order partial derivatives are
  - $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$
  - $\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$
  - $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$
  - $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$

Called mixed partial derivatives

If the function 'f' is nice then the order in mixed derivatives is not important, i.e. $f_{xy} = f_{yx}$

Clairaut's Theorem

Equality of mixed Partial derivatives

Let f be a function of two variables. If f_{xy} and f_{yx} are continuous on some open disk, then $f_{xy} = f_{yx}$ on that disk.

Partial derivatives of functions of more than two variables

- Until now we have only studied partial derivatives of functions of two variables.
- But the concept & computations of partial derivatives of functions of more than two variables are similar. [See example below]

Question 51/921: Find all the second partial derivates of $u = e^{-s}\sin t$.

Question 56/921: Let $u = xye^{y}$ .Verify that the conculsion of Clairaut's Theorem holds

Question 58/921: Let $f(x, y) = x^{2}e^{-ct}$. Find f_{ttt} and f_{txx} .

Question 61/921: Let $u = e^{r\theta}\sin\theta$. Find $\dfrac{\partial^3 u}{\partial r^2 \partial\theta}$

Partial differential equations

Equations involving partial derivatives

Some important examples of partial differential equations are

- $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$ Laplace's equation
- $$\frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2}$$ heat equation
- $$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$ Wave equation

Question 68(e)/921: Determine whether the function

$u = \sin x \cosh y + \cos y \sinh y$ satisfies Laplace's equation.