# 14.2 Limits and Continuity

**Formal Definition of** $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$

Assume the function $f$ is defined at all points within a disk centered at $(x_0, y_0)$, except possibly at $(x_0, y_0)$. We will write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for any given number $\varepsilon > 0$, we can find number $\delta > 0$ such that $f(x,y)$ satisfies

$$|f(x,y) - L| < \varepsilon$$

whenever the distance between $(x,y)$ and $(x_0,y_0)$ satisfies

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

**Informal Definition**

When we move any point $(x,y)$ closer & closer to $(x_0,y_0)$, without actually making it $(x_0,y_0)$,

- Is there a real number $L$ such that the value of $f(x,y)$ becomes closer and closer to $L$.
  - if yes then $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$
  - if not then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist

**Question:** In how many ways can $(x,y)\to(x_0,y_0)$?

**Answer:** Infinite number of ways, e.g. along different lines or curves.

- Here we are concerned mainly with computations of limits
- We begin with learning "How to compute limit by approaching $(x_0, y_0)$ along a given path"

**Question 22/940**: Find the limit of $f(x,y)=\dfrac{x^3y}{2x^6+y^2}$ as $(x,y)\to(0,0)$

along **(a)** the line $y=mx$ **(b)** the parabola $y=kx^2$

Though we will be mostly consider $f(x,y)$, all of our definitions above and study below hold for functions of more variables

## Our three main questions

**1)** When the limit does not exist?

**2)** Techniques for computation of limits?

(Assuming that limit exists)

**3)** Existence of limits question.

Below, we study these questions one by one.

## Q.1: When the limit does not exit?

If $f(x,y)$ has different limits as $(x,y)\to(x_0,y_0)$ along two different paths then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ doesn't exist

- We learn the method through different examples.

**Show that the following limits do not exist:**

**Question 9/908**: $\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{3x^2+y^2}$

Use of paths along X-axis and along $y=x$

**Question 16/908:** $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$

Use of paths along X-axis and along $x=y^4$

## Q.2: Techniques of computing limits $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$

(Assuming limit exists)

**(I)** Plug in values directly e.g. $\lim_{(x,y)\to(1,1)} \frac{xy}{x+y} = \frac{1}{2}$

**(II)** If "direct plugging" in leads to $\left(\frac{0}{0}\right)$ or other indeterminate form then

i. simplify & plug in values

ii. use substitutions

- some other appropriate substitution
- polar coordinates (See Question 38)
- spherical coordinates (See Question 39)

**Compute the following limits**

**Question14/908:** $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}$

**Question 15/908:** $\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

**Question 17/908:** $\lim_{(x,y,z)\to(3,0,1)} e^{-xy} \sin(\pi z / 2)$

**Question 38/909:** $\lim_{(x,y)\to(0,0)} (x^2 + y^2)\ln(x^2 + y^2)$

**Question 39/908:** $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2 + y^2 + z^2}$

## Q.3: Existence question of limits

To get an idea about "How to show existence of limits", we look at the following example

**Example:** Does the limit $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ exist?

**Solution:**

- Computing along X-axis, Y-axis we get $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$
- Computing along the lines $y = kx$ we get $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$
- Computing along parabola $y = x^2$ we get $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

**From above, we expect that limit exists and its value is zero but to ensure that the limit exists we should use formal definition.**

**Question 19/908:** Show that the limit $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$ does not exist.

# Continuity

A function $f(x,y)$ is continuous at $(x_0,y_0)$ if

i. $f(x_0,y_0)$ is defined

ii. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

iii. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Means the graph has no hole or gap at the point $(x_0,y_0)$

- A function $z = f(x,y)$ of two variable is continuous at every point $(x,y)$ in the $xy-$plane is said to be ***continuous everywhere***.
- A composition of continuous function is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

**Continuity on a Set:** Let $R$ denote a subset of the $xy-$plane contained within the domain of a function $f(x,y)$. We say that $f(x,y)$ is ***continuous on*** $R$ provided that for every point $(x_0,y_0)$ in $R$, and for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $f(x,y)$ satisfies

$$|f(x,y) - f(x_0,y_0)| < \varepsilon$$

whenever $(x,y)$ is in $R$ and the distance between $(x,y)$ and $(x_0,y_0)$ satisfies

$$0 \le \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

**Question 29/909**: Determine the set of points at which the function

$F(x,y) = \arctan(x + \sqrt{y})$ is continuous.

**Question 33/909**: Determine the set of points at which the function

$f(x, y, z) = \dfrac{\sqrt{y}}{x^2 - y^2 + z^2}$ is continuous.

**Question 36/941**: Let $f(x, y) = \dfrac{xy}{x^2 + xy + y^2}$ if $(x, y) \neq (0,0)$

and $f(0,0) = 0$. Determine the set of points at which this function is continuous.