14.1 Functions of Several Variables

**Definition:** Let $D \subseteq \mathbb{R}^2$. A function of two variables is a rule that assigns a unique real number $f(x,y)$ to each point $(x,y) \in D$.

The set $D$ is called the domain of $f(x,y)$.

**Definition:** Let $D \subseteq \mathbb{R}^3$. A function of three variables is a rule that assigns a unique real number $f(x,y,z)$ to each point $(x,y,z) \in D$.

The set $D$ is called the domain of $f(x,y,z)$.

**Question 10/898:** For $f(x,y,z) = \ln(25 - x^2 - y^2 - z^2)$

(a) Find $f(2,-2,4)$  
(b) Find domain and range of $f(x,y,z)$  
(c) Sketch domain of $f(x,y,z)$.

**Question 16/898:** Find and sketch domain of the function $f(x,y) = \sqrt{y-x} \ln(y+x)$.

**Exercise:** Find domain of $f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$

**Answer:** $\{D = (x,y) \text{ such that } x + y + 1 \geq 0 \text{ and } x \neq 1\}$
Graph of $f(x, y)$: The graph of $f(x, y)$ is the set of all points $(x, y, z)$ satisfying $z = f(x, y)$.

A surface lying above or below its domain.

Similar to graph of $f(x)$.

**Question 28/899:** Find domain of $f(x, y) = \sqrt{16 - x^2 - 16y^2}$ and sketch the graph of $f(x, y)$. 
**Level curves of \( f(x, y) \):**

Consider the function \( f(x, y) = +\sqrt{x^2 + y^2} \). It represents the upper half of the elliptic cone.

If we fix a value of \( f(x, y) \), say \( f(x, y) = c \), we are basically looking at the cross section of the surface. In this case, we have

\[
+\sqrt{x^2 + y^2} = c \quad \text{or} \quad x^2 + y^2 = c^2
\]

which is a circle of radius \( c \).

In this example, if we have different values of \( c \), we will have cross sections. And we can plot them on the same graph.
We call these curves level curves.

- The projection (on the XY plane) of the curve at height level ‘$k$’ is called a level curve with constant ‘$k$’, that is, the graph of the equation $f(x,y) = k$ (where $k$ is any constant) is called level curve of height $k$.

- A set of some level curves of $f(x,y)$ is called a contour map (or contour plot) of $f(x,y)$. 
**Question 42/931:** Sketch the level curve (or cantor map) $z = k$ for $k = -2, -1, 0, 1, 2$ of the function $z = y \sec x$.

**Level surfaces of $f(x, y, z)$**

- Since the graph of $f(x, y, z)$ would be in 4 dimensional space, therefore, we cannot visualize it easily.
- But we can get an idea from its level surfaces.

**Example:** Let $f(x, y, z) = x^2 + y^2 - z^2$. Then the level surfaces $f(x, y, z) = \gamma$ are given by

![Graph of level surfaces](image-url)
Question 60/900: Describe the level surfaces of the function

\[ f(x, y, z) = x^2 + 3y^2 + 5z^2. \]

Similarly we can define functions of n-variables for n>3

Exercise

For \( f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} x_k \), find \( f(1,1,\ldots,1) \)