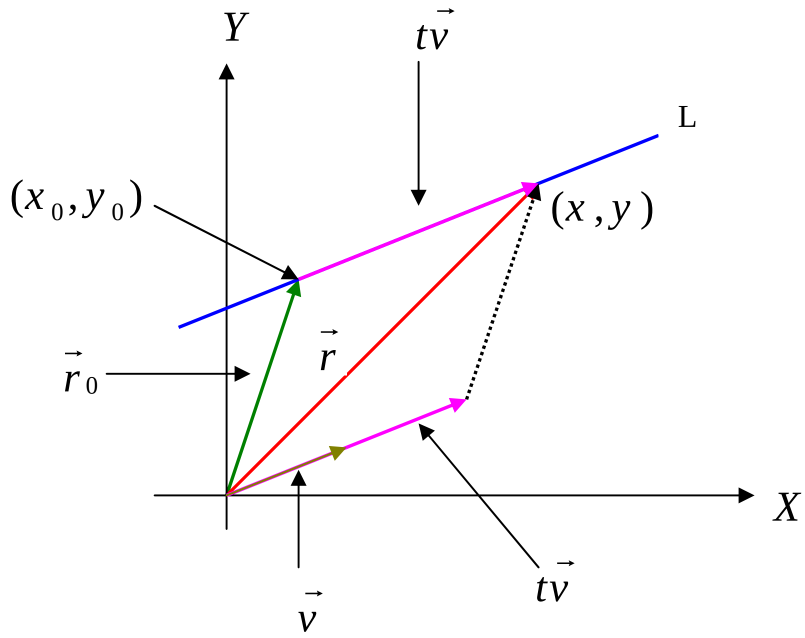
12.5 Equations of Lines and Planes

Parametric Form of Equation of a Line



We define the vectors $\vec{r}, \vec{r_0}$ and $\vec{v}$ as

$$\vec{r} = \langle x, y \rangle, \qquad \vec{r_0} = \langle x_0, y_0 \rangle \qquad \vec{v} = \langle a, b \rangle \qquad \text{[2-space]}$$
$$\vec{r} = \langle x, y, z \rangle, \qquad \vec{r_0} = \langle x_0, y_0, z_0 \rangle \qquad \vec{v} = \langle a, b, c \rangle \qquad \text{[3-space]}$$

Then we have ***vector equation of a line*** in 2-space or 3-space $\vec{r} = \vec{r_0} + t\vec{v}$.

Therefore, a line can be written as

$$\langle x, y \rangle = \langle x_0 + at, y_0 + bt \rangle$$
$$\langle x, y, z \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

or, equivalently, as

$$\langle x, y \rangle = \langle x_0, y_0 \rangle + t \langle a, b \rangle$$
$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

or

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

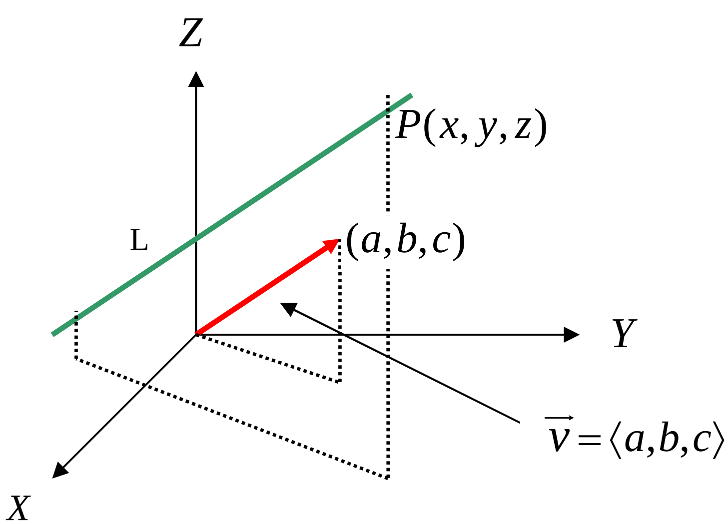
Similar definition in 2-space

Parametric Form

The parametric equation of a line 'L' passing through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = \langle a, b, c \rangle$ is given by

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

Different values of parameter 't' give different points on the line.



To determine parametric equations of a line, we need

* a point on the line
* a vector parallel to the line

If $x = x_0 + at, \ y = y_0 + bt, \ z = z_0 + ct$ are parametric equations of a line, then $\vec{v} = \langle a, b, c \rangle$ is a vector parallel to the given line and the given line passes through the point (x_0, y_0, z_0).

Symmetric form of equation of a line

Similar definition in 2-space

Symmetric form

The symmetric form of the equation of a line 'L' passing through $P_0(x_0, y_0, z_0)$ and parallel to vector $\vec{v} = \langle a, b, c \rangle$ is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

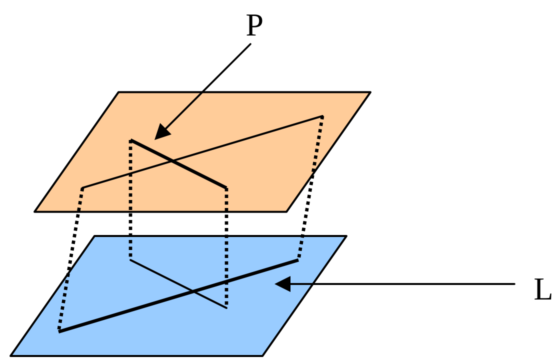
Question 10/830: Find the parametric and symmetric equations for the line through (2, 1, 0) and perpendicular to both $\vec{i} + \vec{j}$ and $\vec{j} + \vec{k}$.

Question 18/830: Find the parametric for the line segment from (10, 3, 1) to (5, 6, -3).

Different types of lines

The lines 'L_1', 'L_2' are called

- parallel if their corresponding vectors $\vec{v}_1, \vec{v}_2$ are parallel i.e. multiple of each other ($\overrightarrow{v_1} = k\overrightarrow{v_2}$ or $\overrightarrow{v_2} = k\overrightarrow{v_1}$ or $\overrightarrow{v_1} \times \overrightarrow{v_2} = \vec{0}$)
- perpendicular if their corresponding vectors $\vec{v}_1, \vec{v}_2$ are orthogonal i.e. $\vec{v}_1 \cdot \vec{v}_2 = 0$
- intersecting if these intersect at a point
- skew if these are neither parallel nor intersecting (therefore do not lie in same plane)



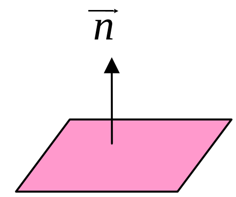
Question 21/830: Determine whether the line L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection:

$$L_1 \; : \; \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$$

$$L_2 \; : \; \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$$

Equation of a plane

A vector perpendicular to a plane is called a ***normal*** to the plane



To determine a plane we require

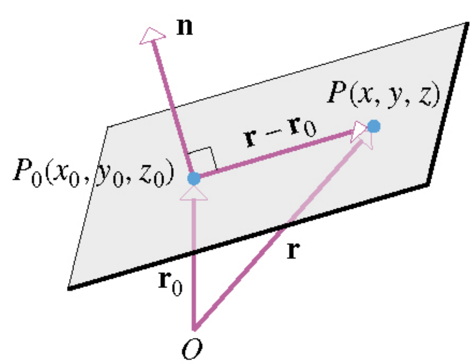
- a point on the plane
- a vector normal to the plane

Vector form

The equation of a plane passing through $P_0(x_0, y_0, z_0)$ and having the normal vector $\vec{v} = \langle a, b, c \rangle$ is

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0, \quad \text{where } \vec{r_0} = \langle x_0, y_0, z_0 \rangle \text{ and } \vec{r} = \langle x, y, z \rangle$$

$$\text{or} \quad \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$



Point-normal form

The equation of a plane passing through $P_0(x_0, y_0, z_0)$ and having the normal vector $\vec{v} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad (*)$$

Can write (*) as

$ax + by + cz = d$

where

$d = ax_0 + by_0 + cz_0$

general form of the equation of a plane

$$ax + by + cz = d \qquad (**)$$

(***a, b, c*** and ***d*** are constants with ***a, b, c*** are not all zero) always gives a plane with normal $\vec{v} = \langle a, b, c \rangle$

Question 26/830: Find the equation of a plane through the point (-2, 8, 10) and perpendicular to the line $x = 1 + t$, $y = 2t$, $z = 4 - 3t$.

Parallel or Perpendicular Planes

Two planes are

- parallel if their normals are parallel
- perpendicular if their normals are orthogonal

Exercise: Determine whether the planes are parallel, perpendicular, or neither:

(a) $$P_1: \quad 2x-8y-6z=3$$
$$P_2: \quad -x+4y+3z=5$$

(b) $$P_1: \quad 3x-2y+z=1$$
$$P_2: \quad 4x+5y-2z=4$$

If $\vec{v}$ is vector parallel to the line ***L*** and $\vec{n}$ is normal to the plane ***P***. Then line ***L*** and plane ***P*** are

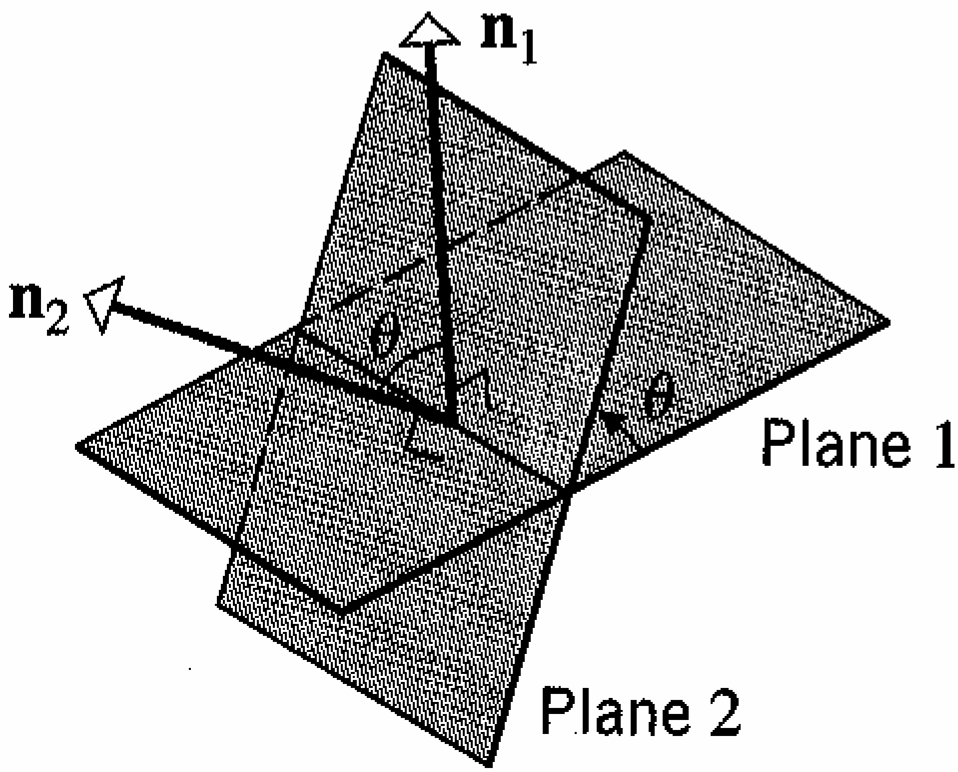
- parallel if $\vec{v}$ and $\vec{n}$ are orthogonal
- perpendicular if $\vec{v}$ and $\vec{n}$ are parallel

Line of intersection of two planes

Can you see how two planes intersect in a line

If P_1, P_2 are two intersecting planes with normals $\vec{n}_1, \vec{n}_2$ then

$\vec{n}_1 \times \vec{n}_2$ is parallel to the line of intersection of P_1, P_2.



Question 37/830: Find the equation of a plane passes through the point (-1, 2, 1) and contains the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$.

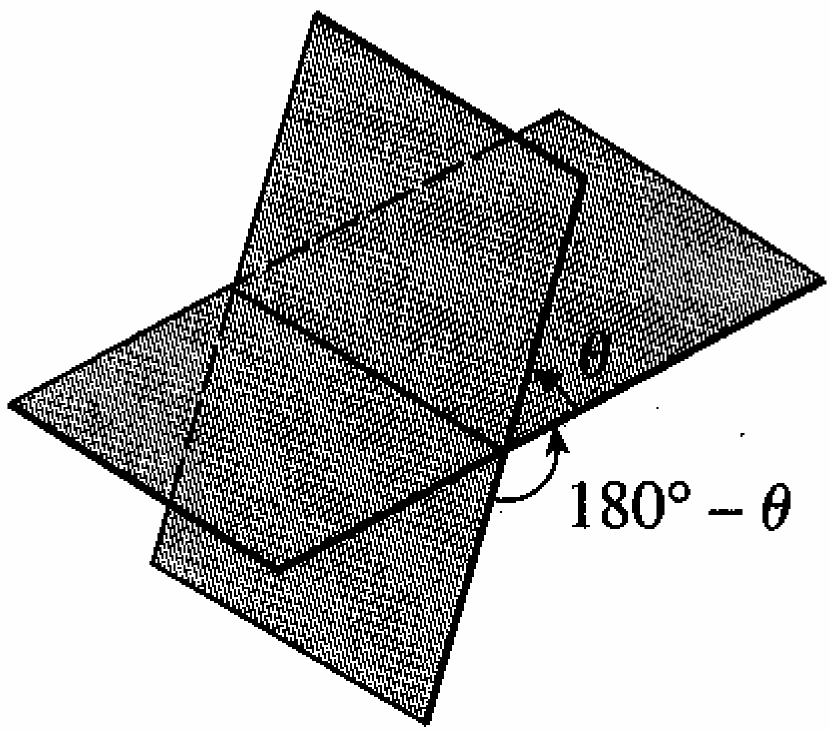
Angle between two intersecting planes

The acute angle θ between planes P_1, P_2 is given by

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\|\vec{n}_2\|}$$

where $\vec{n}_1, \vec{n}_2$ are normals to planes P_1, P_2.

See class explanation



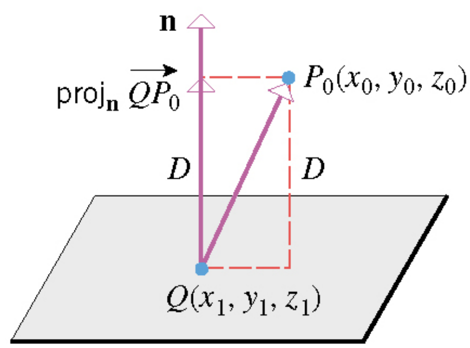
Question 52/831: (a) Find the symmetric equations for the line of intersection of the planes and (b) find the angle between the planes

$$P_1 : x - 2y + z = 1 \quad \text{and} \quad P_2 : 2x + y + z = 1 \,.$$

Distance between a point and a plane

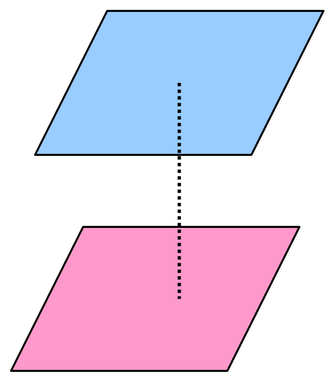
The distance between a point $P_0(x_0, y_0, z_0)$ and the plane $ax+by+cz+d=0$ is given by

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$



The distance ***D*** between parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$$

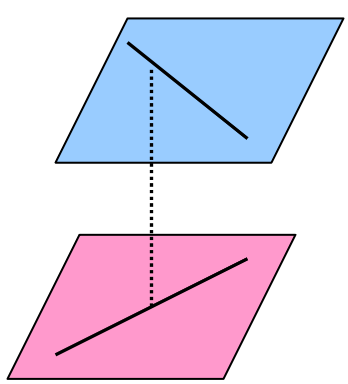


Distance between two skew lines

- Two skew lines L_1, L_2 can be viewed as lying in two parallel planes P_1, P_2
- So the question of finding distance between L_1 and L_2 is equivalent to finding distance between parallel planes P_1 and P_2

What to do

- Find parallel planes P_1, P_2 containing skew lines L_1, L_2
- Find distance between parallel planes P_1, P_2 using above idea.



<u>Question 43/830</u>: Find the direction numbers for the line of intersection of the planes

$$P_1 \quad : \quad x + y + z = 1$$
$$P_2 \quad : \quad x + z = 0$$

<u>Question 48/831</u>: Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them

$$P_1 \quad : \quad 2x - 3y + 4z = 5$$
$$P_2 \quad : \quad x + 6y + 4z = 3$$