12.4 Cross product

- The cross product is defined only for vectors in 3-space, whereas the dot product is defined for vectors in 2-space and 3-space.

Definition

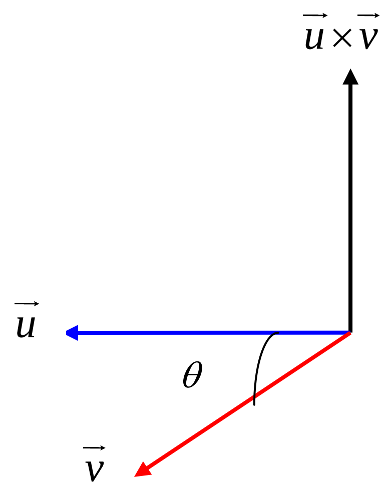
If $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$ then the cross product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \times \vec{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

or, equivalently,

$$\vec{u} \times \vec{v} = \langle u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1 \rangle$$

Result: a vector



Easier way to remember

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Question 6/820: If $\vec{a} = \vec{i} + e^t \vec{j} + e^{-t} \vec{k}$ and $\vec{b} = 2\vec{i} + e^t \vec{j} + e^{-t} \vec{k}$, then find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both $\vec{a}$ and $\vec{b}$.

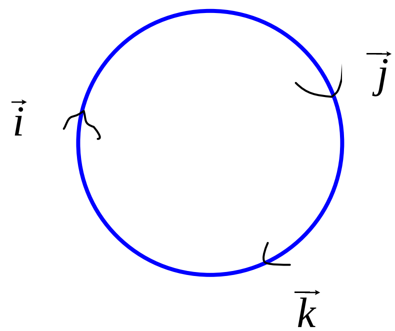
Important fact

$\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$, that is,

$$\vec{u}\cdot(\vec{u}\times\vec{v})=0 \qquad (\vec{u}\times\vec{v}\ \textit{is orthogonal to}\ \vec{u})$$

$$\vec{v}\cdot(\vec{u}\times\vec{v})=0 \qquad (\vec{u}\times\vec{v}\ \textit{is orthogonal to}\ \vec{v})$$

- $\vec{i}\times\vec{j}=\vec{k} \qquad \vec{j}\times\vec{k}=\vec{i} \qquad \vec{k}\times\vec{i}=\vec{j}$
- $\vec{j}\times\vec{i}=-\vec{k} \qquad \vec{k}\times\vec{j}=-\vec{i} \qquad \vec{i}\times\vec{k}=-\vec{j}$
- The associative law does not hold for cross product, for example,
  $\vec{i}\times(\vec{j}\times\vec{j})=\vec{i}\times\vec{0}=\vec{0}$ and $(\vec{i}\times\vec{j})\times\vec{j}=\vec{k}\times\vec{j}=-\vec{i}$
  and so
  $\vec{i}\times(\vec{j}\times\vec{j})\neq(\vec{i}\times\vec{j})\times\vec{j}$



Basic Properties of Vector Product

For any vector $\vec{u}$, $\vec{v}$ and $\vec{w}$ and any scalar k, the following relations hold:

- $\vec{u}\times\vec{v}=-(\vec{v}\times\vec{u})$
- $\vec{u}\times(\vec{v}+\vec{w})=(\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$
- $(\vec{u}+\vec{v})\times\vec{w}=(\vec{u}\times\vec{w})+(\vec{v}\times\vec{w})$
- $k(\vec{u}\times\vec{v})=(k\vec{u})\times\vec{v}=\vec{u}\times(k\vec{v})$
- $\vec{u}\times\vec{0}=\vec{0}\times\vec{u}=\vec{0}$
- $\vec{u}\times\vec{u}=\vec{0}$

Geometric Description of Vector Product

If θ is the angle between $\vec{u}$ and $\vec{v}$ then

$$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$$

assumption
$0 \le \theta \le \pi$

Important fact

$$\vec{u}\times\vec{v} = 0 \quad \Leftrightarrow \quad \vec{u} \text{ and } \vec{v} \text{ are parallel}$$

Exercise: The vectors $\vec{u} = \langle -1,1,1 \rangle$, $\vec{v} = \langle 1,2,3 \rangle$ are parallel?

Answer: Not parallel

<u>Question 15/820</u>: Find two unit vectors orthogonal to both $\langle 1,-1,1 \rangle$ and $\langle 0,4,4 \rangle$.

Application of Cross Product

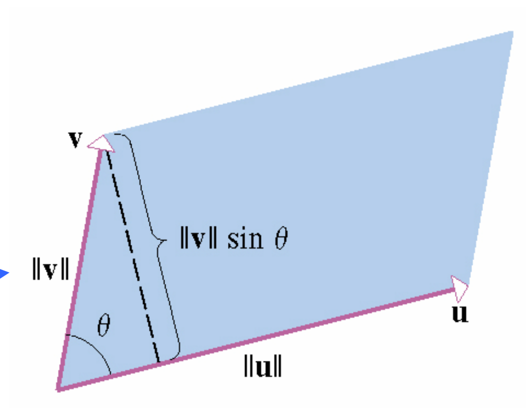
- **Finding area of a parallelogram**

Recall

- The area of a parallelogram determined by vectors $\vec{u}, \vec{v}$ is given by

$$A = \text{base} \times \text{altitude}$$

- If θ is the angle between $\vec{u}, \vec{v}$ then from figure
  - altitude $= \|\vec{v}\| \sin\theta$
  - base $= \|\vec{u}\|$
  - area $= \|\vec{u}\|\|\vec{v}\| \sin\theta = \|\vec{u} \times \vec{v}\|$



The **area of a parallelogram** determined by $\vec{u}$ and $\vec{v}$ is

$$A = \|\vec{u} \times \vec{v}\|$$

The **area of triangle** having adjacent sides given by $\vec{u}$ and $\vec{v}$ is

$$A = \frac{1}{2}\|\vec{u} \times \vec{v}\|$$

Question 24/820: Find the area of the parallelogram with vertices $K(1,2,3)$, $L(1,3,6)$, $M(3,8,6)$ and $N(3,7,3)$.

Scalar Triple Product

The product $\vec{u}\cdot(\vec{v}\times\vec{w})$ is called

scalar triple product of $\vec{u}$, $\vec{v}$ and $\vec{w}$

Result: a scalar

Efficient way of computing

If $\vec{u}=\langle u_1,u_2,u_3\rangle$, $\vec{v}=\langle v_1,v_2,v_3\rangle$ and $\vec{w}=\langle w_1,w_2,w_3\rangle$ then

$$\vec{u}\cdot(\vec{v}\times\vec{w})=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Important property

$$\vec{u}\cdot(\vec{v}\times\vec{w})=(\vec{u}\times\vec{v})\cdot\vec{w} \text{ and}$$

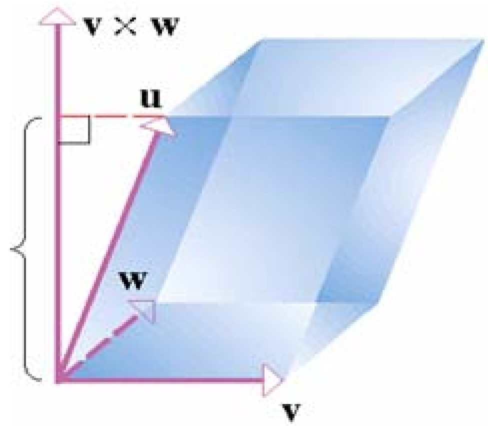
$$\vec{u}\cdot(\vec{v}\times\vec{w})=\vec{w}\cdot(\vec{u}\times\vec{v})=\vec{v}\cdot(\vec{w}\times\vec{u})$$

Application of Scalar Triple Product

- **Finding volume of a parallelepiped**

Recall

- The volume of a parallelepiped determined by vectors $\vec{u}, \vec{v}, \vec{w}$ is given by $V = \text{area of base} \times \text{altitude}$
- If θ is the angle between $\vec{u}$ and $\vec{v} \times \vec{w}$ then from figure
  - o altitude $= \|\vec{u}\| |\cos\theta|$
  - o area of base $= \|\vec{v} \times \vec{w}\|$
  - o $V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$



The **volume of a parallelepiped** determined by $\vec{u}, \vec{v}$ and $\vec{w}$ is

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$$

Important property

$\vec{u}, \vec{v}$ and $\vec{w}$ are coplanar

$\Leftrightarrow \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = 0$

Question 32/820: Find the volume of the parallelepiped with adjacent edges *PQ, PR* and *PS,* where *P*(0,1,2), *Q*(2,4,5), *R*(-1,0,1), *S*(6,-1,4).

Question 34/820: Use the scalar triple product to determine whether the points *P*(1,0,1), *Q*(2,4,6), *R*(3,-1,2), and *S*(6,2,8) lie in the same plane.