12.3 Dot Product

Multiplication of Vectors

There are two types of multiplication of vectors.

- Dot Product: The multiplication of two vectors produces a scalar
- Cross Product: The multiplication of two vectors produces a vector

Dot product

Given two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

Their dot product is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad \textbf{(*)}$$

Similar definition for vectors in 2-space

Result is a scalar

Properties of dot product

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vector and k is a scalar. Then

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v} = \vec{u} \cdot (k\vec{v})$
4. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$
5. $\vec{0} \cdot \vec{v} = 0$

Question 8/812: Find $\vec{a} \cdot \vec{b}$ if $\vec{a} = 4\vec{j} - 3\vec{k}$ and $\vec{b} = 2\vec{i} + 4\vec{j} + 6\vec{k}$.

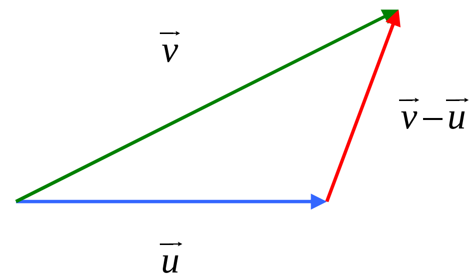
Geometric interpretation in terms of angle between vectors

If θ $(0 \le \theta \le \pi)$ is the angle between $\vec{u}$ and $\vec{v}$ then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

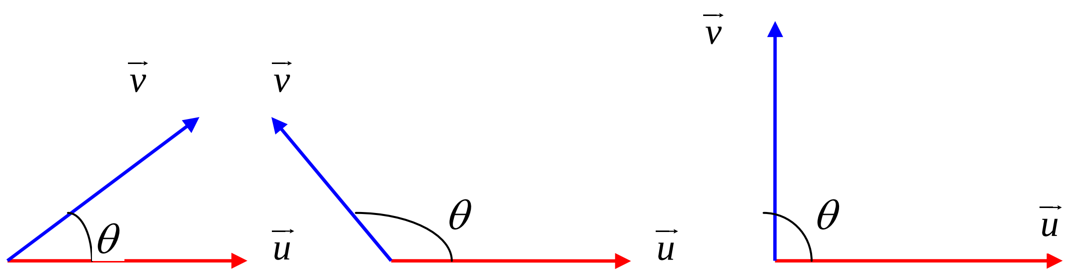
In other words, the dot product is also defined as

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$



Observation

- $\vec{u}\cdot\vec{v} > 0 \Rightarrow$ angle between $\vec{u}$ and $\vec{v}$ is acute
- $\vec{u}\cdot\vec{v} < 0 \Rightarrow$ angle between $\vec{u}$ and $\vec{v}$ is obtuse
- $\vec{u}\cdot\vec{v} = 0 \Rightarrow$ $\vec{u}$ and $\vec{v}$ are orthogonal



$\vec{u}\cdot\vec{v} > 0$ $\quad$ $\vec{u}\cdot\vec{v} < 0$ $\quad$ $\vec{u}\cdot\vec{v} = 0$

Question 19/812: Find the angle between the vectors $\vec{a} = \vec{j} + \vec{k}$ and $\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$.

Question 26/813: For what values of b are the vector $\langle -6, b, 2\rangle$ and $\langle b, b^2, b\rangle$ orthogonal?

Direction Angles and Direction Cosines of a Vector

Given a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$

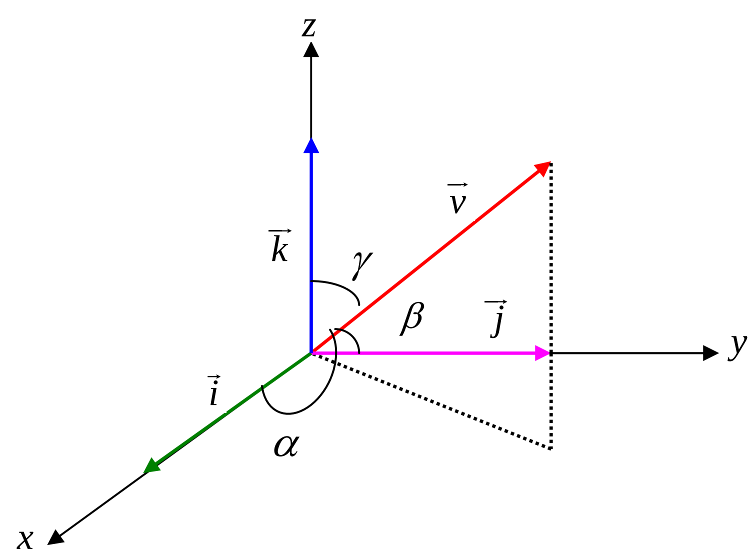
- Angles α, β, γ between $\vec{v}$ and vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ (i.e., between $\vec{v}$ and X-axis, Y-axis, Z-axis, respectively), are called ***direction angles***.
- The cosines $\cos\alpha, \cos\beta, \cos\gamma$ of direction angles are called ***direction cosines*** of $\vec{v}$.

The ***direction cosines*** of $\vec{v} = \langle v_1, v_2, v_3 \rangle$ are

$$\cos\alpha = \frac{\vec{v}\cdot\mathbf{i}}{\|\vec{v}\|\|\mathbf{i}\|} = \frac{v_1}{\|\vec{v}\|}$$

$$\cos\beta = \frac{\vec{v}\cdot\mathbf{j}}{\|\vec{v}\|\|\mathbf{j}\|} = \frac{v_2}{\|\vec{v}\|}$$

$$\cos\gamma = \frac{\vec{v}\cdot\mathbf{k}}{\|\vec{v}\|\|\mathbf{k}\|} = \frac{v_3}{\|\vec{v}\|}$$

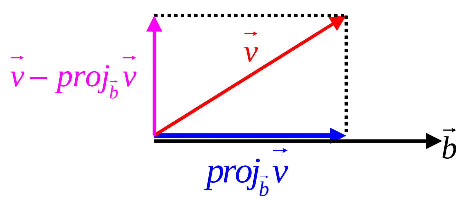


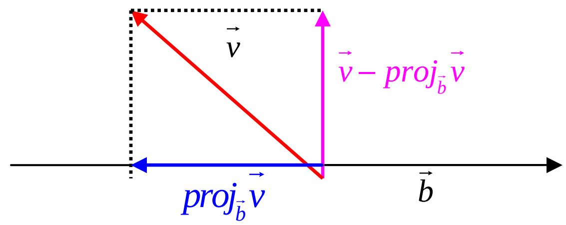
<u>Question 32/813</u>: Find the direction cosines and direction angles of the vector $2\vec{i} - \vec{j} + 2\vec{k}$.

Projection of a Vector on another Vector

Q. What is projection of $\vec{v}$ on $\vec{b}$ (geometrically)?

A. Roughly speaking shadow of $\vec{v}$ on $\vec{b}$ (see figures below & explanation in class)

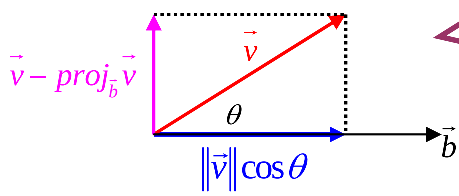




Using dot product to find projection of a vector on another

Let θ be angle between $\vec{v}$ and $\vec{b}$.

Then



- Since $\|\vec{v}\|\cos\theta = \dfrac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|}$
- Hence $proj_{\vec{b}}\vec{v} = \left(\dfrac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|}\right)\left(\dfrac{\vec{b}}{\|\vec{b}\|}\right)$

The projection of $\vec{v}$ on $\vec{b}$ is given by $proj_{\vec{b}}\vec{v} = \left(\dfrac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$

$proj_{\vec{b}}\vec{v} = \left(\dfrac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|}\right)\left(\dfrac{\vec{b}}{\|\vec{b}\|}\right)$ = vector project (or vector component) of $\vec{v}$ on (or along) $\vec{b}$.

$\vec{v} - proj_{\vec{b}}\vec{v}$ = vector component of $\vec{v}$ orthogonal to $\vec{b}$

$comp_{\vec{b}}\vec{v} = \dfrac{\vec{a}\cdot\vec{v}}{||\vec{b}||}$ = scalar projection of $\vec{v}$ on $\vec{b}$

Question 40/813: Find the scalar and vector projection of $\vec{b} = \vec{i} + 6\vec{j} - 2\vec{k}$ onto $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$.

Question 52/813: Find the angle between a diagonal of a cube and a diagonal of one of its faces.