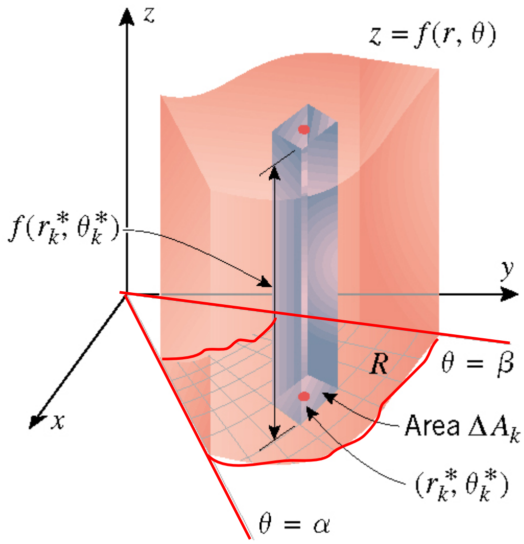
15.3 Double integrals in polar coordinates

Evaluating Double Integrals in Polar Coordinates

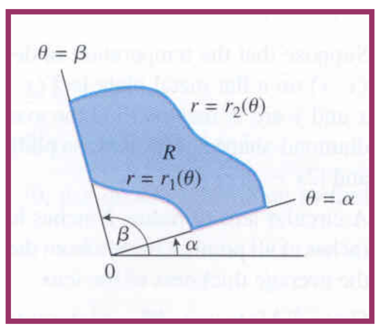


Simple polar region

A region defined by

- $\{(r,\theta):\alpha \le \theta \le \beta \text{ and } 0 \le r_1(\theta) \le r \le r_2(\theta)\}$

  where $\beta - \alpha \le 2\pi$



Formula for evaluating double integral in polar coordinates

Given $f(r,\theta)$ over a simple region R given by

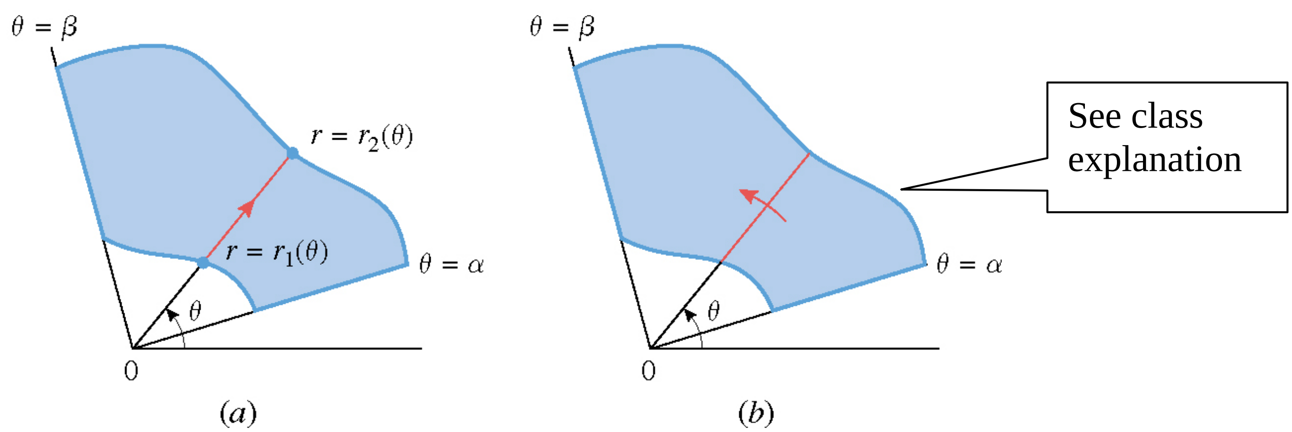
$$\{(r,\theta):\alpha \le \theta \le \beta \text{ and } 0 \le r_1(\theta) \le r \le r_2(\theta)\}.$$

Then

$$\iint_R f(r,\theta)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

Note: Area element dA

- in rectangular coordinates: $dA = dxdy$
- in polar coordinates: $dA = rdrd\theta$



Find limit of Integrals for polar region R

Step 1. Draw a radial line from the origin through the region R at a fixed angle θ. This line intersects the region on the inner boundary curve $r = r_1(\theta)$ and outer boundary curve $r = r_2(\theta)$. Then $r = r_1(\theta)$ is the lower limit and $r = r_2(\theta)$ is the upper limit of the inner integral.

Step 2. Rotate a ray along the polar X-axis one revolution counter-clockwise about the origin. The smallest angle at which this ray intersects the region R is $\theta = \alpha$ and the largest angle is $\theta = \beta$. $\theta = \alpha$ is the lower limit and $\theta = \beta$ is the upper limit of the outer integral.

Question 6/1034: Evaluate $\int_0^{\pi/2}\int_0^{\cos\theta} r^3 drd\theta$.

Application

- If $f(r,\theta) \geq 0$ then $\iint_R f(r,\theta)\,dA$ gives volume under surface

  $z = f(r,\theta)$ over the region R.

- $\iint_R dA$ gives area of the region R.

Question 10/1034: Find the area of the region inside the circle $x^2 + y^2 = 4$ and to the right of the line $x = 1$.

Converting Double Integrals into Polar Coordinates

Some double integrals are easier to evaluate in polar coordinates.

Especially,

- If the region R consists of a circle or a part of the circle. i.e contains an equation of the form $x^2 + y^2 = a^2$
- If the integrand contains expressions of the form $x^2 + y^2 = a^2$.

How to convert?

Use

- $x = r\cos\theta$, $y = r\sin\theta$ $\Rightarrow$ $x^2 + y^2 = r^2$

Question 22/1035: Use polar coordinates to evaluate $\iint_R 2y\,dA$, where R is the region in the first quadrant bounded above by the circle $(x-1)^2 + y^2 = 1$ and bounded by the line $y = x$.

Question 29/1035: Evaluate $\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1+x^2+y^2}}\,dxdy$ by converting to polar coordinates.

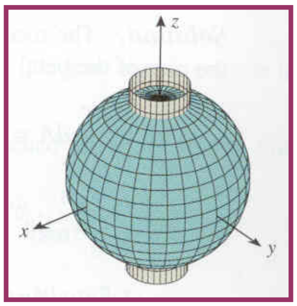
Exercise Find $I = \iint_R e^{x^2+y^2} dA$ over the circle $x^2 + y^2 = 1$

Answer: $I = \int_0^{2\pi} \int_0^r re^{r^2} dr d\theta = \pi(e-1)$

Question 13/1034: Find the volume of solid that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 4$

Hint:

- Figure:



- In rectangular coordinates the volume is given by $V = \iint_R \sqrt{9 - x^2 - y^2} dA$ where R is the region inside the circle $x^2 + y^2 = 4$.

Not a nice integration

- Do we get easier integration in polar coordinates?

Solve all solved examples given in the book and Questions 1—12, 17—30 and 34—35.