15.2 Double Integrals over Non-Rectangular Regions

Evaluating Double Integrals over Non-Rectangular Regions

We will consider two types of regions

Type I region

A region bounded by

- the lines $x = a,\ x = b$

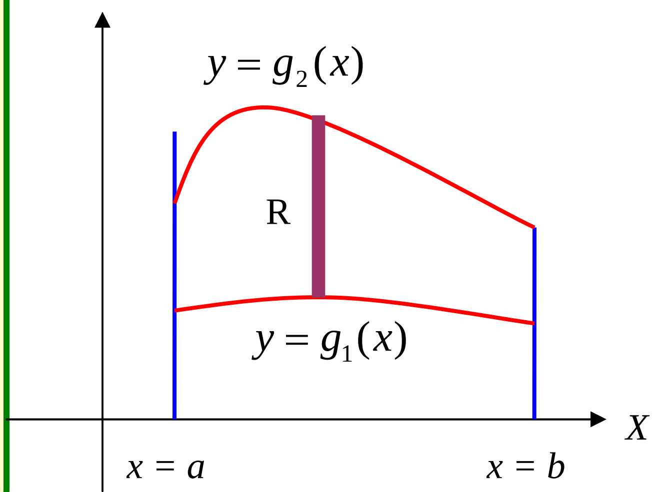
and

- the curves $y = g_1(x),\ \ y = g_2(x)$

  with $g_1(x) \le g_2(x) \quad \forall x \in [a,b]$

Type I region is

$$R_V = \{(x,y): a \le x \le b,\ \ g_1(x) \le y \le g_2(x)$$



Type II region

A region bounded by

- the lines $y = c,\ y = d$

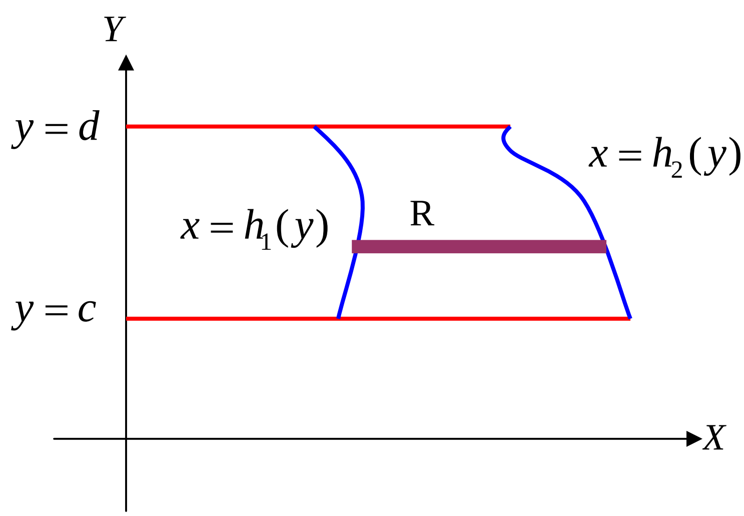
and

- the curves $x = h_1(y),\ \ x = h_2(y)$

  with $h_1(y) \le h_2(y) \quad \forall y \in [c,d]$

Type II region is

$$R_H = \{(x,y): c \le y \le d, h_1(y) \le x \le h_2(y)\}$$



Double integrals over both regions evaluated as iterated integrals

If R is type I then

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

If R is type II then

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Find limit of Integrals for type I region R

Step 1. Draw a vertical line through the region R at an arbitrary fixed value *x*. This line intersects the region below at the curve $y = g_1(x)$ and above at the curve $y = g_2(x)$. Then $y = g_1(x)$ is the lower limit and $y = g_1(x)$ is the upper limit of the inner integral.

Step 2. Move the line left and then right. Leftmost position where the line intersects the region R is $x = a$ which is lower limit of the outer integral. Rightmost position where the line intersects the region R is $x = b$ which is the upper limit of the outer integral.

Find limit of Integrats for type II region R

Step 1. Draw a horizontal line through the region R at an arbitrary fixed value *y*. This line intersects the region left at the curve $y = h_1(x)$ and right at the curve $y = h_2(x)$. Then $y = h_1(x)$ is the lower limit and $y = h_1(x)$ is the upper limit of the inner integral.

Step 2. Move the line down and then up. Lowest position where the line intersects the region R is $y = c$ which is lower limit of the outer integral. Highest position where the line intersects the region R is $y = d$ which is the upper limit of the outer integral.

Important points to note:

- Outer limits always constant
- The idea of using a vertical line as explained in class
- Sometimes need to break into more integrals (see example 15.2.3)

Question 9/1027: Evaluate $\int_0^1 \int_0^x y\sqrt{x^2 - y^2}\,dydx$.

Question 15/1027: Evaluate the double integral $\iint_R (3x - 2y)\,dA$ in two ways using iterated integrals: (a) viewing R as a type I region, and (b) viewing R as a type II region, where R is the region enclosed by the circle $x^2 + y^2 = 1$.

Question 19/1027: Evaluate the double integral $\iint_R xy\,dA$; R is the region enclosed by $y = \sqrt{x}$, $y = 6 - x$ and $y = 0$.

Question 25/1027: Use double integral to find the area of the plane region enclosed by the curves $y = \sin x$ and $y = \cos x$, for $0 \le x \le \pi/4$.

Question 32/1028: Use double integral to find the volume of the solid in the first octant bounded above by the paraboloid $z = x^2 + 3y^2$, below by the plane $z = 0$, and laterally by $y = x^2$ and $y = x$.

Question 48/1028: Evaluate $\int_0^2 \int_{y/2}^1 \cos(x^2)\,dxdy$ by reversing the order of integration.