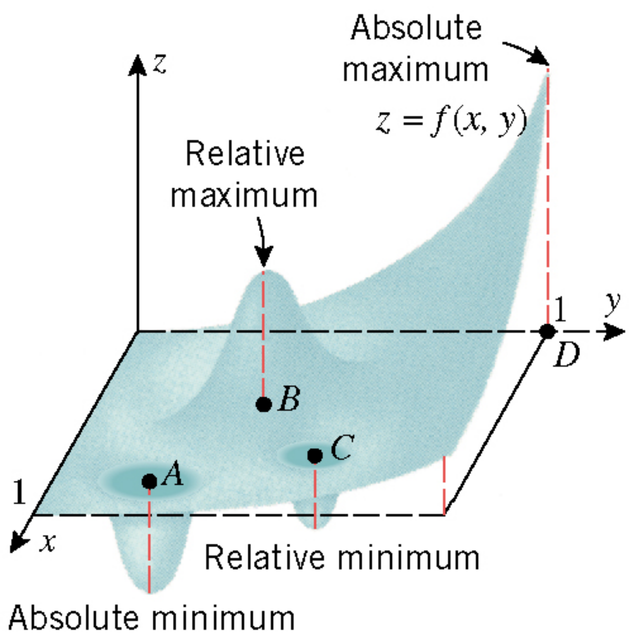
14.8 Maxima & Minima of Functions of Two Variables

$f(x,y)$ has a relative minimum at (x_0,y_0) if $f(x,y) \ge f(x_0,y_0)$ for all points near (x_0,y_0)

$f(x,y)$ has a relative maximum at (x_0,y_0) if $f(x,y) \le f(x_0,y_0)$ for all points near (x_0,y_0)



$f(x,y)$ has an absolute minimum at (x_0,y_0) if $f(x,y) \ge f(x_0,y_0)$ for all the points in domain of $f(x,y)$

$f(x,y)$ has an absolute maximum at (x_0,y_0) if $f(x,y) \le f(x_0,y_0)$ for all the points in domain of $f(x,y)$

If *f* has either relative (absolute) max. or min., we call *f* has relative (absolute) extremum.

Main question:

How to find relative or absolute extrema?

Finding Relative Extrema of *z=f(x,y)*

STEP 1: Find critical points of $f(x,y)$

These are all possible points for extrema

A point (x_0, y_0) is a critical point of $f(x,y)$ if

- $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

OR

- $f_x(x_0, y_0)$ and/or $f_y(x_0, y_0)$ do not exist.

i.e. all points where partial derivatives are zero or do not exist.

STEP 2: Determine whether or not $f(x,y)$ has extrema at critical points

Find $D = f_{xx} f_{yy} - \left(f_{xy}\right)^2$ at critical points

Second partials test

1. If $D(\text{at C.P.}) > 0$ and $f_{xx}(\text{at C.P.}) > 0$ then f has a relative minimum at C.P.
2. If $D(\text{at C.P.}) > 0$ and $f_{xx}(\text{at C.P.}) < 0$ then f has a relative maximum at C.P..
3. If $D(\text{at C.P.}) < 0$ then f has neither relative minimum or relative maximum. In this case C.P. are called saddle points.
4. If $D(\text{at C.P.}) = 0$ then test fails (Use some other methods).

Question 17/1000: Locate all relative maximum, relative minimum and saddle point, if any, of the function $f(x,y) = e^x \sin y$

Finding Absolute Extrema of *z=f(x,y)* on Closed & Bounded Sets

- Note: The existence of absolute extrema of *f(x,y)* is guaranteed over closed & bounded domains.

Extreme value theorem

If $f(x,y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum on R.

- Se of all points inside the set A and all boundary points is called a closed set.
- A set A is called bounded if the distance between any two points in A is finite.

These absolute extrema can occur on the boundary of R or at an interior (critical) point

Procedure for finding absolute extrema on closed & bounded set *R*

1. Find all critical points that lie inside *R*.
2. Find all critical points on the boundary (By using Math 101).
3. Find $f(x,y)$ at these critical points (inside and on the boundary) and at corners (if any). The largest of these values is absolute maximum & the smallest is absolute minimum.

Question 30/1000: Find absolute extrema of the function $f(x,y) = xe^{y} - x^{2} - e^{y}$ on the closed and bounded region R with vertices (0,0), (0,1), (2,1) and (2,0).

Question 34/1000: Find three positive numbers whose sum is 27 and such that the Sum of their squares is as small as possible.

Exercise (Applied problem)

A rectangular box with no top is to be constructed to have a volume $V = 12\,ft^3$. The cost per square foot of the material to be used is SR.4 for the bottom, SR.3 for two of the opposite sides, and SR.2 for remaining pair of opposite sides. Find the dimensions of the box that will minimize the cost.

Hints

- Let x, y, z be sides. Then we have
  - area of base = xy
  - two sides of area = xz
  - two sides of area = yz
- Hence, the cost is given by $C = 4xy + 3(2xz) + 2(2yz)$
- Using $xyz = 12$ we can write C as function of two variables
  - i.e. $C(x, y) = 4xy + \dfrac{72}{y} + \dfrac{48}{x}$ is to be minimized for $x > 0,\ y > 0$.
- Note $C(x, y)$ is to be minimized for $x > 0,\ y > 0$. That means it is not an extrema problem over closed & bounded region.
  - Hence, there is no guarantee of absolute minimum (by theory)
    - But from physical situation we have assurance of minimum.
- So we can continue looking for minimum using the method of relative extrema.
- Complete from here. Answer: $x = 2, y = 3, z = 2$

Solve all solved Examples given in book and Questions 1—4, 9—20, 27—39.