14.6 Directional Derivatives and Gradients

We have done partial derivatives

- f_x: rate of change of 'f' in x-direction
- f_y: rate of change of 'f' in y-direction

In this section, we will see the directional derivatives

- rate of change of 'f' in any given direction

Before defining the directional derivative, we study the gradient of a function of two or more variables

Gradient of a function

For a function $f(x, y, z)$, the gradient vector of $f(x, y, z)$ is defined as

Also called gradient of 'f'

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$\nabla f(x, y, z) = f_x(x, y, z)\vec{i} + f_y(x, y, z)\vec{j} + f(x, y, z)\vec{k}$$

Similar definition can be defined for functions of two or more variables.

∇f is a vector

Properties of ∇f : (Same as derivatives)

If f and g are differentiable, then

- $\nabla(f \pm g) = \nabla f \pm \nabla g$
- $\nabla(cf) = c\nabla f$ (where c is any constant)
- $\nabla(fg) = f\nabla g + g\nabla f$
- $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$
- $\nabla(f^n) = nf^{n-1}\nabla f$

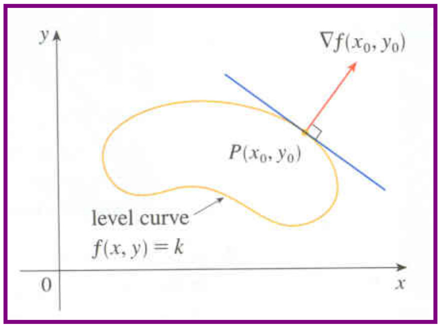
Important Fact About Gradient

Given a surface $z = f(x, y)$.

- Of course the direction $\nabla f\left(x_0, y_0\right)$ is important
- How to determine this direction from the contour map of $f(x, y)$

Recall that for $z = f(x, y)$ we can find the level curve $f(x, y) = k$ that passes through the point (x_0, y_0)

- Let $z = f(x, y)$ be a surface and $f(x, y) = k$ be the level curve that passes through (x_0, y_0).
- Then $\nabla f\left(x_0, y_0\right)$ is normal to the level curve $f(x, y) = k$



- Let $f(x,y,z)$ be differentiable at (x_0,y_0,z_0) and $\nabla(x_0,y_0,z_0)\neq\vec{0}$.
- Then $\nabla f\left(x_0,y_0,z_0\right)$ is normal to the level surface of f through (x_0,y_0,z_0)

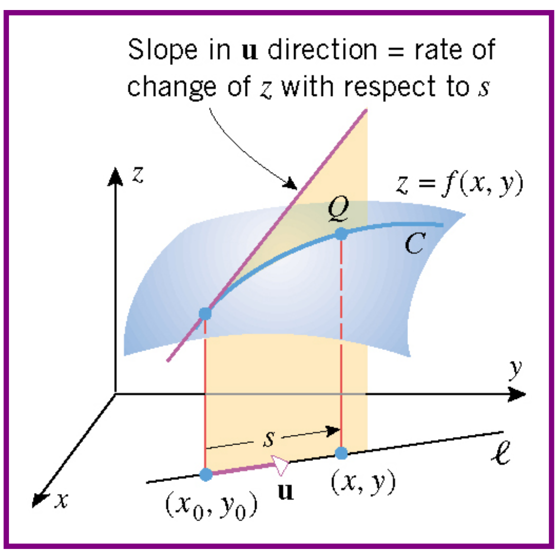
Question 39/983: Find the gradient of $f(x,y,z)=y\ln(x+y+z)$ at $P(-3,4,0)$.

Question 44/983: Sketch the level curve of $f(x,y)=x^2-y^2$ that passes through $P(2,-1)$ and draw the gradient vector at P.

What is Directional Derivative

in the Direction of a Vector $\vec{\mathbf{u}}$ at $\left(x_0,y_0\right)$?

- Slope of surface $z=f(x,y)$ at $\left(x_0,y_0\right)$ in the direction of $\vec{\mathbf{u}}$
- Rate of change of $z=f(x,y)$ at $\left(x_0,y_0\right)$ in the direction of $\vec{\mathbf{u}}$



Main question:

How to compute directional derivatives?

How to Compute Directional Derivative

The directional derivative of $f(x,y)$ in the direction of **unit vector** $\vec{\mathbf{u}} = <u_1, u_2>$ is given by

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)u_1 + f_y(x,y)u_2$$

Assumption

f(x,y) differentiable

The directional derivative of $f(x,y,z)$ in the direction of **unit vector** $\vec{\mathbf{u}} = <u_1, u_2, u_3>$ is given by

$$D_{\mathbf{u}} f(x,y,z) = f_x(x,y,z)u_1 + f_y(x,y,z)u_2 + f_z(x,y,z)u_3$$

Similar formulas for more variables

Directional Derivative in terms of Gradient

The directional derivative of $f(x,y,z)$ in the direction of **unit vector** $\vec{\mathbf{u}} = <u_1, u_2, u_3>$ is given by

$$D_{\mathbf{u}} f = \nabla f \cdot \vec{\mathbf{u}}$$

Exercise: If $f(x,y) = x\cos y$.

(a) Find the gradient of f.

(b) Find directional derivative of f at $(1,0)$ in the direction of $\vec{\mathbf{v}} = \langle 2,1 \rangle$.

Question 17/982: Find the directional derivative of $f(x,y,z) = \dfrac{z-x}{z+y}$ at $P(1,0,-3)$ in the direction of $\vec{a} = -6\vec{i} + 3\vec{j} - 2\vec{k}$.

Question 26/982: Let $f(x,y) = \dfrac{y}{x+y}$. Find a unit vector $\vec{u}$ for which $D_{\vec{u}} f(2,3) = 0$.

Exercise: Find directional derivative of $f(x,y) = e^{xy}$ at $(-2,0)$ in the direction of $\theta = \dfrac{\pi}{3}$.

Exercise: Find directional derivative of $f(x,y,z) = x^3 z + y^3 z^2 - xyz$ in the direction of $\vec{\mathbf{v}} = \langle -1,0,3 \rangle$.

Note to use unit vector to find directional derivatives

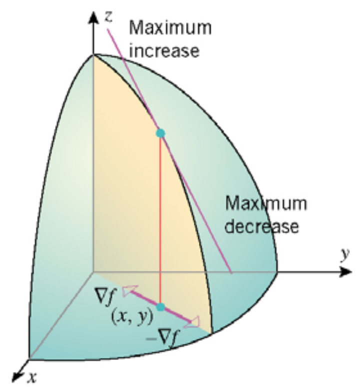
Question 45/983: Find a unit vector $\vec{u}$ that is normal at $P(1,-2)$ to the level curve of $f(x,y) = 4x^2 y$ through P

Important Fact

Gradient vector determines the maximum/minimum rate of change of a function

Let 'f' be a function of 2 or 3 variables.

- The maximum value (minimum value) of the directional derivative of 'f' occurs in direction (opposite to that) of gradient vector ∇f .
- Hence, the maximum value (minimum value) of the directional derivative of 'f' (i.e. maximum (minimum) rate of change of 'f') is '$|\nabla f|$' ('$-|\nabla f|$').



Why?

Since $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$

Question 54/983: Find a unit vector in the direction in which $f(x,y,z) = \tan^{-1}\left(\dfrac{x}{y+z}\right)$ increases most rapidly at $P(4,2,2)$; and find the rate of change of f in that direction.

Exercise: Find the direction in which $f(x,y,z) = 4e^{xy}\cos z$ decreases most rapidly at $P(0,1,\frac{\pi}{4})$. Find the rate of change at $P(0,1,\frac{\pi}{4})$ in that direction.

Solve all solved Examples given in the book and Questions 1—30 and 33—62.