14.5 The Chain Rules

Recall chain rule for functions of one variable

If $y = f(x)$ and $x = g(t)$ then $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$

- Here we study chain rule for functions of more variables
- It has different versions

Chain Rule Case-I

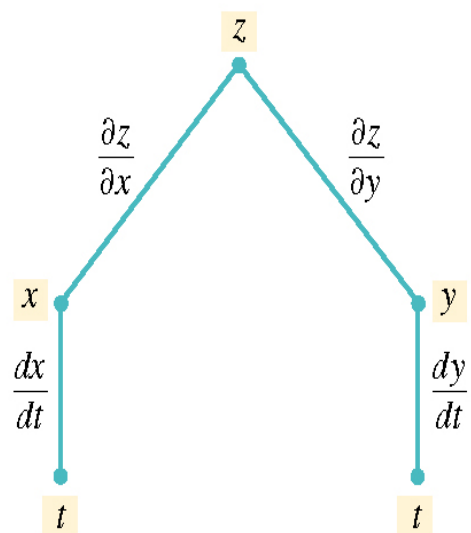
If $z = f(x, y)$ and $x = g(t)$, $y = h(t)$ then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

If $w = f(x, y, z)$ and $x = g(t)$, $y = h(t)$

$z = k(t)$ then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$



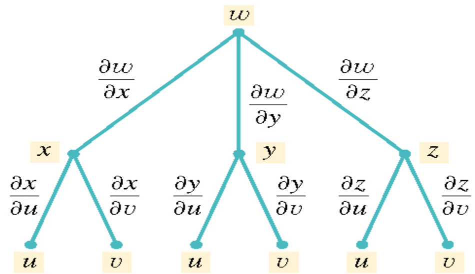
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Question 8/970: If $w = \ln(2x^2 - 2y + 4z^3)$ and $x = t^{1/2}$, $y = t, z = t^3$ then find dw/dt by using an appropriate form of the chain rule.

Chain Rule **Case-II**

If $z = f(x,y)$ and $x = g(u,v),\ y = h(u,v)$

then
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

If $w = f(x,y,z)$ and $x = g(u,v),\ y = h(u,v),$

$z = k(u,v)$ then
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

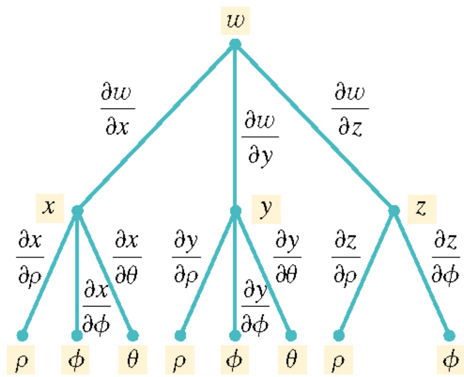
Question 26/971: Let $w = 3xy^2z^3,\ y = 3x^2 + 2,\ z = \sqrt{x-1}$. Find dw/dx.

Use $\dfrac{dw}{dx} = \dfrac{\partial w}{\partial y}\cdot\dfrac{dy}{dx} + \dfrac{\partial w}{\partial z}\cdot\dfrac{dz}{dx}$.

Similar to above logic you should be able to develop the chain rule for different situations.

If $w = f(x, y, z)$ is a function of $x = x(\rho, \phi, \theta), y = (\rho, \phi, \theta)$ and $z = (\rho, \phi)$, then the relevant formula is

$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho} \quad \text{and} \quad \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}$$



$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}$$

Exercise Let $w = f(x, y, z)$ and $x = g(t, u, v),\ y = h(t, u, v),\ z = k(t, u, v)$.

Write down the chain rule formula for $\frac{\partial w}{\partial t}$.

Answer: $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t}$

Implicit Differentiation

- Given $F(x,y)=C$

Defining y implicitly as function of x

- Differentiating w.r.t. x we get

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-F_x}{F_y}$$

If $F(x,y)=C$ implicitly defines y as a function of x then

$$\frac{dy}{dx}=\frac{-F_x}{F_y} \qquad (\text{if } F_y \neq 0)$$

Question 33/971: Find $\dfrac{dy}{dx}$ from $e^{xy}+ye^{y}=1$.

Defining z implicitly as function of x and y

- Consider $F(x,y,z)=C$ (*)
  - Differentiate (using chain rule) Eq. (*) w.r.t. x and show that

$$\frac{\partial z}{\partial x}=\frac{-F_x}{F_z}$$

See Q.35 in the book

  - Differentiate (using chain rule) Eq. (*) w.r.t. y and show that

$$\frac{\partial z}{\partial y}=\frac{-F_y}{F_z}$$

See Q.36 in the book

Question 40/971: If $e^{xy}\cos yz-e^{yz}\sin xz+2=0$. Find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

Question 43/971: Two sides of a triangle have lengths $a=4$ cm and $b=3$ cm, but are increasing at the rate of 1 cm/s. If the area of the triangle remains constant, at what rate is the angle θ between a and b changing when $\theta=\pi/6$?

Solve all solved examples given in the book and Questions 1—45.