14.3 Partial derivatives

Formal Definition

- The partial derivative of $f(x,y)$ w.r.t. 'x' at the point (x_0, y_0) is denoted as $\dfrac{\partial f}{\partial x}$ and defined as

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

Rate of change in 'x' direction

- The partial derivative of $f(x,y)$ w.r.t. 'y' at the point (x_0, y_0) is denoted as $\dfrac{\partial f}{\partial y}$ and defined as

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Rate of change in 'y' direction

Informal Definition $\dfrac{\partial f}{\partial x}$

Ordinary derivative of $f(x,y)$ w.r.t. 'x'

- keeping 'y' as constant

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Ordinary derivative of $f(x,y)$ w.r.t. 'y'

- keeping 'x' as constant

Notation

Consider $z = f(x, y)$.

- The partial derivatives w.r.t. 'x' and 'y' are denoted respectively by

$$f_x, \frac{\partial f}{\partial x}, \frac{\partial z}{\partial x} \qquad \text{and} \qquad f_y, \frac{\partial f}{\partial y}, \frac{\partial z}{\partial y}$$

- The partial derivatives at the point (x_0, y_0) are

$$f_x(x_0, y_0), \left.\frac{\partial f}{\partial x}\right|_{x=x_0,\ y=y_0}, \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)} \quad \text{and} \quad f_y(x_0, y_0), \left.\frac{\partial f}{\partial y}\right|_{x=x_0,\ y=y_0}, \left.\frac{\partial z}{\partial y}\right|_{(x_0, y_0)}$$

Question 2/949: Let $z = e^{2x} \sin y$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \left.\frac{\partial z}{\partial y}\right|_{(0,y)}$ and $\left.\frac{\partial z}{\partial x}\right|_{(\ln 2,\, 0)}$

Question 16/950: Consider $z = \frac{x^2 y^3}{\sqrt{x+y}}$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Question 21/950: Consider $x^2 \sin(2y - 5z) = 1 + y\cos(6xz)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

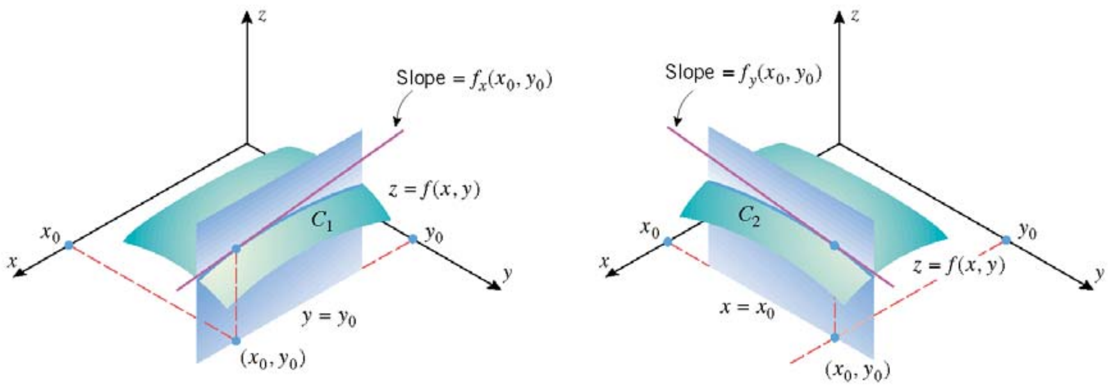
Question 32/950: Consider $f(x, y, z) = \cosh(\sqrt{z})\sinh^2(x^2 yz)$. Find f_x, f_y and f_z.

Implicit partial differentiation

- Learn through an example

Question 60/952: If $\ln(2x^2 + y - z^3 + 3w) = z$ then calculate $\partial w/\partial x$, $\partial w/\partial y$ and $\partial w/\partial z$.

Interpretation of partial derivatives of z = f(x,y)



$\dfrac{\partial f}{\partial x}(x_0, y_0)$

- **rate of change** of *f(x,y)* in **x-direction**
- **slope** of surface *z=f(x,y)* in **x-direction**

$\dfrac{\partial f}{\partial y}(x_0, y_0)$

- **rate of change** of *f(x,y)* in **y-direction**
- **slope** of surface *z=f(x,y)* in **y-direction**

Question 4(a)/949: Find the slope of the surface $f(x, y) = xe^{-y} + 5y$ in the *X*-direction at the point (3,0).

Question 6(b)/949: Find the rate of change of $z = (x + y)^{-1}$ w.r.t. y at the point (-2,4) with x held fixed.

Eercise: Find the slope of the tangent line at $(-1,1,5)$ to the curve of intersection of the surface $z = x^2 + 4y^2$ and

(a) the plane $x = -1$ — i.e. y-direction

(b) the plane $y = 1$ — i.e. x-direction

Question 41/951: A point moves along the intersection of the elliptic paraboloid $z = x^2 + 3y^2$ and the plane $y = 1$. At what rate is z changing with x when the point is at (2,1,7)?

Higher order partial derivatives

- Since $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are also functions of *x & y*, so we can differentiate them further
- For *z=f(x,y)*, the four second order partial derivatives are
  - $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$
  - $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$
  - $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$
  - $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$

Called mixed partial derivatives (f_{xy} and f_{yx})

If the function '*f*' is nice then the order in mixed derivatives is not important, i.e. $f_{xy} = f_{yx}$

Equality of mixed Partial derivatives

Let f be a function of two variables. If f_{xy} and f_{yx} are continuous on some open disk, then $f_{xy} = f_{yx}$ on that disk.

Eercise: The 1st order partial derivative of $f(x,y) = \cos 2x - x^2 e^{5y} + 3y^2$ are $f_x = -2\sin 2x - 2xe^{5y}$ and $f_y = -5x^2 e^{5y} + 6y$. Find all 2nd order partial derivatives.

Exercise: Let $f(x,y) = xe^{-x^2 y^2}$. Verify that $f_{xy} = f_{yx}$.

Partial derivatives of functions of more than two variables

- Until now we have only studied partial derivatives of functions of two variables.
- But the concept & computations of partial derivatives of functions of more than two variables are similar. [See example below]

Question 81/952: Calculate $f_{xyy}(0,1)$, $f_{xxx}(0,1)$, and $f_{yyxx}(0,1)$ for $f(x,y,z) = y^3 e^{-5x}$.

Exercise Calculate f_{xxyzz} for $f(x,y,z) = z^3 y^2 \ln x$.

Partial differential equations

Equations involving partial derivatives

Some important examples of partial differential equations are

- $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ — Laplace's equation
- $\frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2}$ — heat equation
- $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ — Wave equation

Question 85(a)/952: Show that $z = x^2 - y^2 + 2xy$ satisfies Laplace's equation.

Solve all solved examples given in the book and Questions 1—6, 11—38, 40—85.