14.2 Limits and Continuity

Formal Definition of $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$

Assume the function f is defined at all points within a disk centered at (x_0, y_0), except possibly at (x_0, y_0). We will write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \mathrm{L}$$

if for any given number $\varepsilon > 0$, we can find number $\delta > 0$ such that $f(x,y)$ satisfies

$$| f(x,y) - L | < \varepsilon$$

whenever the distance between (x,y) and (x_0, y_0) satisfies $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

Informal Definition

When we move any point (x, y) closer & closer to (x_0, y_0), without actually making it (x_0, y_0),

- Is there a real number L such that the value of $f(x, y)$ becomes closer and closer to L.
  - if yes then $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$
  - if not then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist

Question: In how many ways can $(x, y) \to (x_0, y_0)$?

Answer: Infinite number of ways, e.g. along different lines or curves.

- Here we are concerned mainly with computations of limits
- We begin with learning "How to compute limit by approaching (x_0, y_0) along a given path"

Question 26(a)/940: Find the limit of $f(x,y)=\dfrac{x^3y}{2x^6+y^2}$ as $(x,y)\to(0,0)$ along

(a) the line $y=mx$ **(b)** the parabola $y=kx^2$

Though we will be mostly consider $f(x,y)$, all of our definitions above and study below hold for functions of more variables

Our three main questions

1) When the limit does not exist?

2) Techniques for computation of limits?

(Assuming that limit exists)

3) Existence of limits question.

Below, we study these questions one by one.

Q.1: When the limit does not exit?

If $f(x,y)$ has different limits as $(x,y)\to(x_0,y_0)$ along two different paths then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ doesn't exist

- We learn the method through different examples.

Show that the following limits do not exist:

Question 7(b)/940: $\lim_{(x,y)\to(0,0)} \frac{x+y}{x+y^2}$

Use of paths along coordinate axes, lines etc

Question: $\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$

infinite limit

Question 27(b)/940: $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^4+z^4}$

Use of different paths e.g. lines in 3-dim. $x=at,\ \ y=bt,\ \ z=ct$ and curve $x=t^2,\ \ y=t,\ z=t$

Question: Use polar coordinates to show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

Question 20/940: Use spherical coordinates to show that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{\sin\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}$$ does not exist.

Q.2: Techniques of computing limits $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$

(Assuming limit exists)

(I) Plug in values directly e.g. $\lim_{(x,y)\to(1,1)} \frac{xy}{x+y} = \frac{1}{2}$

(II) If "direct plugging" in leads to $\left(\frac{0}{0}\right)$ or other indeterminate form then

i. simplify & plug in values [See example 14.2.8]

ii. use substitutions

- polar coordinates [See example 14.2.9]
- spherical coordinates [See example 14.2.10]
- or some other appropriate substitution [see Qs.9-12 in the book]

Compute the following limits

Question14/940: $\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$

Question 21/940: $\lim_{(x,y)\to(0,0)} y\ln\left(x^2 + y^2\right)$

Question24/940: $\lim_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\frac{1}{x^2 + y^2 + z^2}\right]$

Q.3: Existence question of limits

To get an idea about "How to show existence of limits", we look at the following example

Example: Does the limit $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ exist?

Solution:

- Computing along X-axis, Y-axis we get $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$
- Computing along the lines $y = kx$ we get $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$
- Computing along parabola $y = x^2$ we get $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

From above, we expect that limit exists and its value is zero but to ensure that the limit exists we should use formal definition.

Since showing existence "using definition" is not part of our syllabus so we don't proceed further. Our purpose was only to get an idea about "how to handle existence of limits questions".

Question 9/940: [In some cases, we can reduce problem to single variable case]

Find the limit $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ as $(x,y) \to (0,0)$ if it exists?

Continuity

A function $f(x,y)$ is continuous at (x_0,y_0) if

i. $f(x_0,y_0)$ is defined

ii. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

iii. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Means the graph has no hole or gap at the point (x_0,y_0)

- A function $z = f(x,y)$ of two variable is continuous at every point (x,y) in the $xy-$plane is said to be ***continuous everywhere***.
- A composition of continuous function is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

Continuity on a Set: Let R denote a subset of the $xy-$plane contained within the domain of a function $f(x,y)$. We say that $f(x,y)$ is ***continuous on*** R provided that for every point (x_0,y_0) in R, and for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $f(x,y)$ satisfies

$$|f(x,y) - f(x_0,y_0)| < \varepsilon$$

whenever (x,y) is in R and the distance between (x,y) and (x_0,y_0) satisfies

$$0 \le \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

Question 30/941: Let $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$ if $(x,y) \ne (0,0)$. Define $f(0,0)$ so that f is continuous at $(0,0)$.

Question 44/941: Find the region on which the function

$$f(x,y,z) = \sin\sqrt{x^2+y^2+3z^2}$$ is continuous.

Solve all Solved examples given in the book and Questions 1—24, 26—44.