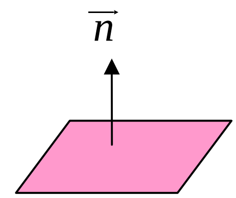
12.6 Planes in 3-space

Equation of a plane

A vector perpendicular to a plane is called a ***normal*** to the plane



To determine a plane we require

- ✱ a point on the plane
- ✱ a vector normal to the plane

The equation of a plane passing through $P_0(x_0, y_0, z_0)$ and having the normal vector $\vec{v} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad (*)$$

Point-normal form

Can write (*) as

$$ax + by + cz = d$$

where

$$d = ax_0 + by_0 + cz_0$$

$$ax + by + cz = d \qquad (**)$$

general form of the equation of a plane

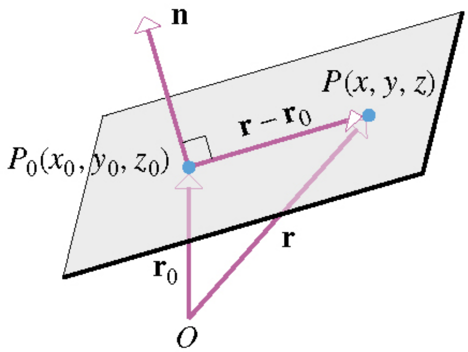
(***a, b, c*** and ***d*** are constants with ***a, b, c*** are not all zero) always gives a plane with normal $\vec{v} = \langle a, b, c \rangle$

The equation of a plane passing through $P_0(x_0, y_0, z_0)$ and having the normal vector $\vec{v} = \langle a,b,c \rangle$ is

$$\langle a,b,c\rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Vector form

or $\vec{n}\cdot(\vec{r}-\vec{r_0}) = 0,$ where $\vec{r_0} = \langle x_0, y_0, z_0\rangle$ and $\vec{r} = \langle x, y, z\rangle$



Exercise Find the equation of the plane through the point $P(5,-2,4)$ and having normal $\vec{n} = \langle 4,2,3 \rangle$.

Question 12/836: Find the equation of the plane through the points $P(3,2,1)$, $Q(2,1,-1)$ and $R(-1,3,2)$.

Exercise Find the equation of the plane that passes through $P_1(6,0,-2)$ and contains the line $x = 4 - 2t,\ y = 3 + 5t,\ z = 7 + 4t$.

Hint: Find two points P_2, P_3 on the line and use idea of previous example. Answer: $33x + 10y + 4z = 190$

Parallel or perpendicular planes

Two planes are

- parallel if their normals are parallel
- perpendicular if their normals are orthogonal

Question 13/836: Determine whether the planes are parallel, perpendicular, or neither:

(a) $P_1: \quad 2x-8y-6z=3$
$P_2: \quad -x+4y+3z=5$

(b) $P_1: \quad 3x-2y+z=1$
$P_2: \quad 4x+5y-2z=4$

If $\vec{v}$ is vector parallel to the line ***L*** and $\vec{n}$ is normal to the plane ***P***. Then line ***L*** and plane ***P*** are

- parallel if $\vec{v}$ and $\vec{n}$ are orthogonal
- perpendicular if $\vec{v}$ and $\vec{n}$ are parallel

Question 16(c)/836: Determine whether the line $x=t, \ \ y=1-t, \ \ z=2+t$ and the plane $x+y+z=1$ are parallel, perpendicular, or neither.

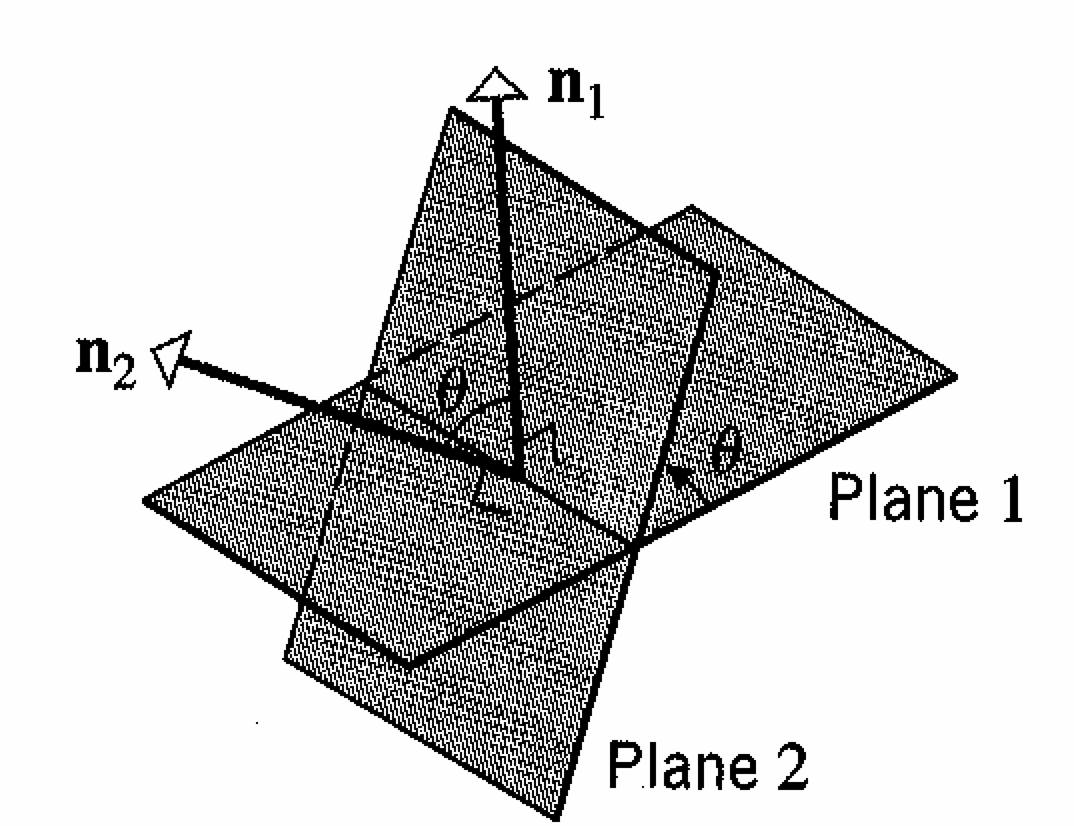
Question 18(a)/836: Determine whether the line $x=3t, \ \ y=5t, \ \ z=-t$ and the plane $2x-y+z+1=0$ intersect; if so, find the coordinates of the intersection.

Line of intersection of two planes

Can you see how two planes intersect in a line

If P_1, P_2 are two intersecting planes with normals $\vec{n}_1, \vec{n}_2$ then

$\vec{n}_1 \times \vec{n}_2$ is parallel to the line of intersection of P_1, P_2.



Exercise: Find equation of the plane through $P(5,-2,4)$ that is parallel to the plane $3x + y - 6z + 8 = 0$

Question 26/837: Find equation of the plane through the points $P_1(-2,1,4)$, $P_2(1,0,3)$ that is perpendicular to the plane $4x - y + 3z = 2$.

Question 27/837: Find equation of the plane through $P(-1,2,-5)$ that is perpendicular to the planes $2x - y + z = 1$ and $x + y - 2z = 3$.

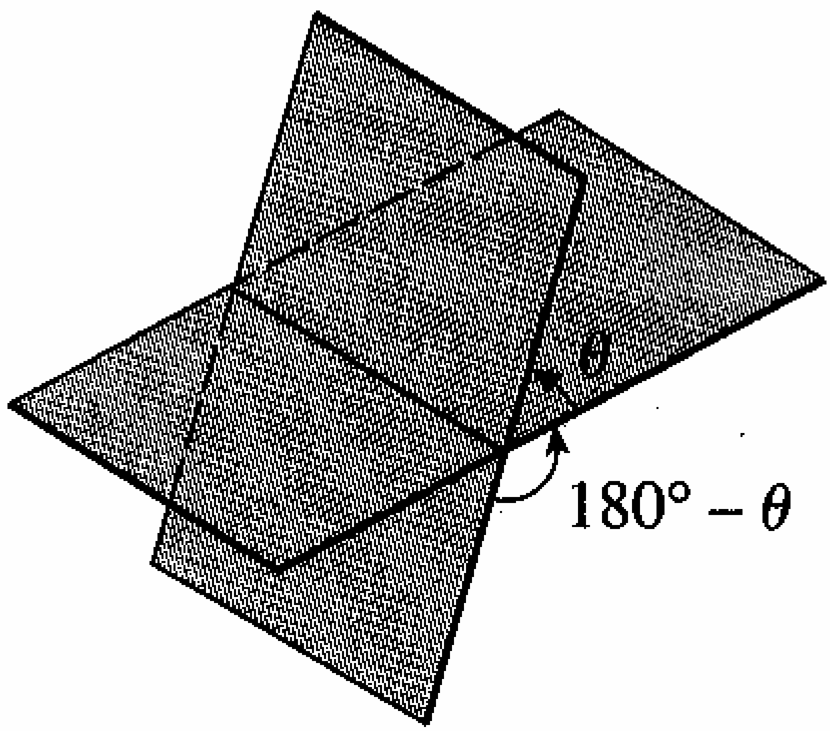
Angle between two intersecting planes

The acute angle θ between planes P_1, P_2 is given by

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \|\vec{n}_2\|}$$

where $\vec{n}_1, \vec{n}_2$ are normals to planes P_1, P_2.

See class explanation



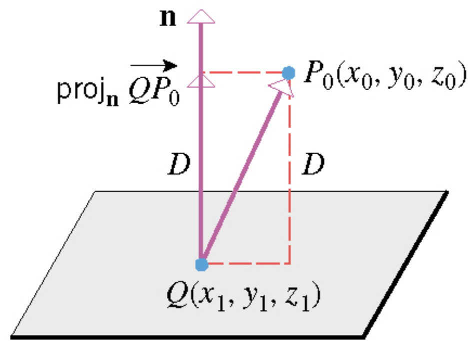
Question 20/837: Find the acute angle of intersection of the planes

$$P_1: x + 2y - 2z = 5 \quad \text{and} \quad P_2: \; 6x - 3y + 2z = 8 \; .$$

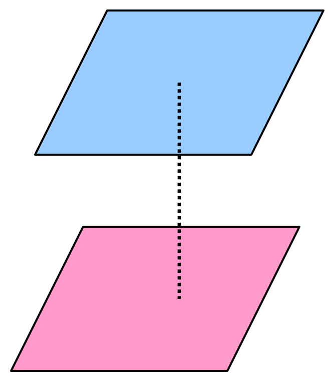
Distance between a point and a plane

The distance between a point $P_0(x_0, y_0, z_0)$ and the plane $ax+by+cz+d=0$ is given by

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$



The distance D between parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is $D = \dfrac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$



Question 39/837: Find the distance between the point $P(1,-2,3)$ and the plane $2x-2y+z=4$.

Question 41/838: Show that the following planes are parallel and find the distance between them.

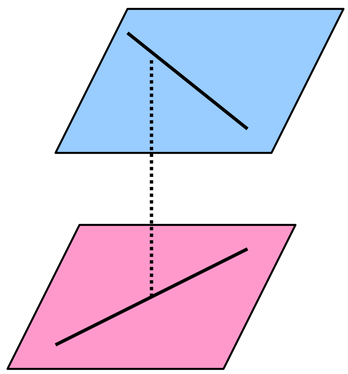
$$P_1 \quad : \quad -2x+y+z=0$$
$$P_2 \quad : \quad 6x-3y-3z=5$$

Distance between two skew lines

- Two skew lines L_1, L_2 can be viewed as lying in two parallel planes P_1, P_2
- So the question of finding distance between L_1 and L_2 is equivalent to finding distance between parallel planes P_1 and P_2

What to do

- Find parallel planes P_1, P_2 containing skew lines L_1, L_2
- Find distance between parallel planes P_1, P_2 using above idea.



Question 43/837: Show that the following lines are skew and find the distance between them.

$$L_1 \quad : \quad x = 1 + 7t, \;\; y = 3 + t, \;\; z = 5 - 3t$$
$$L_2 \quad : \quad x = 4 - t, \;\; y = 6, \;\; z = 7 + 2t$$

Solve all solved examples given in the book and questions 1—47.