12.5 Parametric Equations of Lines

Parametric Form of Equation of a Line

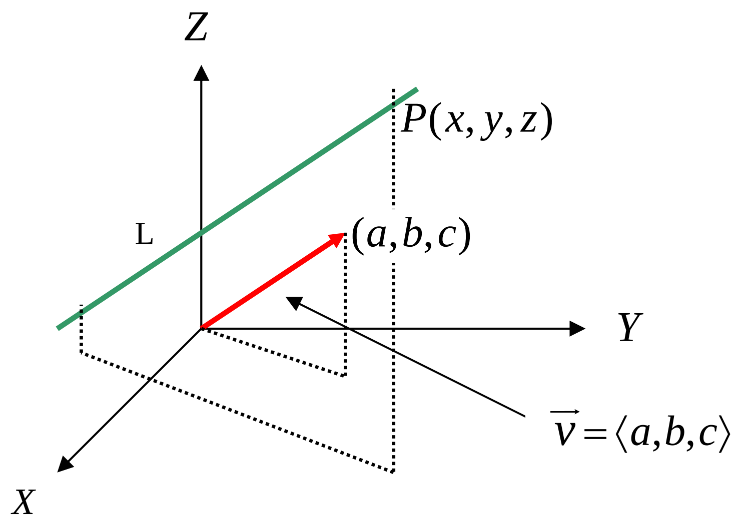
Similar definition in 2-space

Parametric Form

The parametric equation of a line 'L' passing through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = \langle a, b, c \rangle$ is given by

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

Different values of parameter 't' give different points on the line.



To determine parametric equations of a line, we need

- a point on the line
- a vector parallel to the line

If $x = x_0 + at, \; y = y_0 + bt, \; z = z_0 + ct$ are parametric equations of a line, then $\vec{v} = \langle a, b, c \rangle$ is a vector parallel to the given line and the given line passes through the point (x_0, y_0, z_0).

The parametric equations of a line can also be written in vector notation as

$$\langle x, y, z\rangle = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle \text{ or, } \langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

We define the vectors $\overrightarrow{r}, \overrightarrow{r_0}$ and $\overrightarrow{v}$ as

$$\overrightarrow{r} = \langle x, y, z\rangle, \quad \overrightarrow{r_0} = \langle x_0, y_0, z_0\rangle, \quad \overrightarrow{v} = \langle a, b, c\rangle$$

Then the *vector equation of a line* in 3-space is $\overrightarrow{r} = \overrightarrow{r_0} + t\overrightarrow{v}$.

Exercise Find the parametric equation of the line 'L' that passes through the point $P_0(5,1,3)$ and is parallel to $\vec{v} = \langle 1, 4, -2\rangle$. Find two points on the line.

Exercise Find a vector parallel to the line

$$x = 1 + 4t,\ y = 2 + 5t,\ z = -3 - 7t$$

Question 4(b)/829: Find the parametric equations of a line through the points $P_1(-1,3,5)$ and $P_2(-1,3,2)$.

Question 12/829 Find the parametric equations of line passing through $(0,3)$ and parallel to the line $x = -5 + t,\ y = 1 - 2t$.

Question 13/829: Find the parametric equations for the line that is tangent to the circle $x^2 + y^2 = 25$ at the point $(3,-4)$.

Different types of lines

The lines 'L_1', 'L_2' are called

- parallel if their corresponding vectors $\vec{v}_1, \vec{v}_2$ are parallel i.e. multiple of each other ($\overrightarrow{v_1} = k\overrightarrow{v_2}$ or $\overrightarrow{v_2} = k\overrightarrow{v_1}$ or $\overrightarrow{v_1} \times \overrightarrow{v_2} = \vec{0}$)
- perpendicular if their corresponding vectors $\vec{v}_1, \vec{v}_2$ are orthogonal i.e. $\vec{v}_1 \cdot \vec{v}_2 = 0$
- intersecting if these intersect at a point
- skew if these are neither parallel nor intersecting (therefore do not lie in same plane)

Question 29/830 Are the following lines parallel

$$L_1: \ x = 3 - 2t, \ y = 4 + t, \ z = 6 - t$$

$$L_2: \ x = 5 - 4t, \ y = -2 + 2t, \ z = 7 - 2t$$

Finding intersection of two line (given in parametric form)

Given two lines

$$L_1: \quad x = x_0 + a_0 t,\ y = y_0 + b_0 t,\ z = z_0 + c_0 t$$

$$L_2: \quad x = x_1 + a_1 s,\ y = y_1 + b_1 s,\ z = z_1 + c_1 s$$

- L_1 and L_2 intersect if there exist values of s, t such that

$$x_0 + a_0 t = x_1 + a_1 s \qquad (1)$$
$$y_0 + b_0 t = y_1 + b_1 s \qquad (2)$$
$$z_0 + c_0 t = z_1 + c_1 s \qquad (3)$$

Important to use different parameters

- We solve any two equations to get value of t, s
- Then we put these values in the 3rd equation:
  - if 3rd equation is satisfied then lines intersect
  - if 3rd equation is not satisfied then lines do not intersect

Question 25/829 Find the points of intersection of

$$L_1: x = 2 + t,\ y = 2 + 3t,\ z = 3 + t$$

$$L_2: x = 2 + s,\ y = 3 + 4s,\ z = 4 + 2s$$

Question 28/830 Show that the following lines are skew.

$$L_1: x = 2 + 8t,\ y = 6 - 8t,\ z = 10t$$

$$L_2: x = 3 + 8t,\ y = 5 - 3t,\ z = 6 + t$$

Intersection of a line with other curves, planes and surfaces

Learn the method through examples.

Question 24/829 Where does the line

$$L:\ x = 2 - t,\ y = 3t,\ z = -1 + 2t \qquad (1)$$

intersect the plane

$$2y + 3z = 6? \qquad (*)$$

To solve, we need to answer the following question:

"Is there a value of t for which x, y, z satisfy the equation

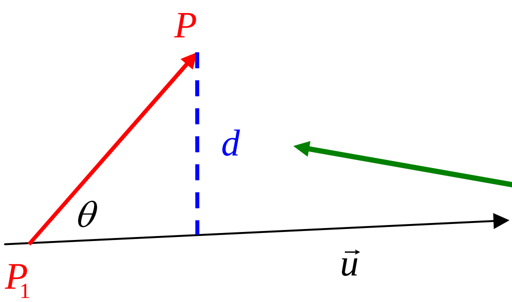
$2y + 3z = 6$ of the plane"

Exercise Find the points of intersection of

$$L:\ x = 1 + 3t,\ y = -1 + t,\ z = 2 - 2t$$

with the plane $2x - 3y + z = 6$. Answer: $(-2, -2, 4)$

Finding distance between a point and a line in 3-space



Given a point P and a line L.

- Let P_1 be a point on L and $\vec{u}$ a vector along L
- If d is the distance between P and L then

$$d = \left\|\overrightarrow{P_1P}\right\| \sin\theta = \frac{\left\|\overrightarrow{P_1P} \times \vec{u}\right\|}{\left\|\vec{u}\right\|}$$

To find distance '*d*' from a point P to a line L

- Choose a point P_1 on L
- Find a vector $\vec{u}$ along L
- Then $$d = \frac{\left\|\overrightarrow{P_1P} \times \vec{u}\right\|}{\left\|\vec{u}\right\|}$$

Exercise Find the distance between $P(-2,1,1)$ and the line

$L:\ x = 3 - t,\ y = t,\ z = 1 + 2t$

Finding distance between two parallel lines

- Take a point P on one line
- Find distance of P from other line by the method shown above.

Question 39/830: Show that the following lines are parallel,

$$L_1: x = 2 - t,\ y = 2t,\ z = 1 + t$$

$$L_2: x = 1 + 2t,\ y = 3 - 4t,\ z = 5 - 2t$$

and find the distance between them.

Question 47/830 Find the equation of line that passes through $P(0,2,1)$ and intersects the line $L:\ x = 2t,\ y = 1 - t,\ z = 2 + t$ at right angle.

Solve all the solved examples given in the book and 1—40, 43—48.