12.3 Dot Product; Projections

Multiplication of Vectors

There are two types of multiplication of vectors.

- Dot Product: The multiplication of two vectors produces a scalar
- Cross Product: The multiplication of two vectors produces a vector

Dot product

Given two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

Their dot product is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad (*)$$

Similar definition for vectors in 2-space

Result is a scalar

Properties of dot product

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vector and k is a scalar. Then

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v} = \vec{u} \cdot (k\vec{v})$
4. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$
5. $\vec{0} \cdot \vec{v} = 0$

Exercise 12.3.1 Find the dot product of

a. $\vec{u} = \langle 5, -8, 0 \rangle$, $\vec{v} = \langle 1, 2, 0 \rangle$ Answer: -11

b. $\vec{u} = \langle 0, 3, -7 \rangle$, $\vec{v} = \langle 2, 3, 1 \rangle$ Answer: 2

Geometric interpretation in terms of angle between vectors

If θ is the angle between $\vec{u}$ and $\vec{v}$ then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

assumption

$0 \le \theta \le \pi$

- Proof: Using law of cosines
- See book or Math002 notes

In other words, the dot product is also defined as

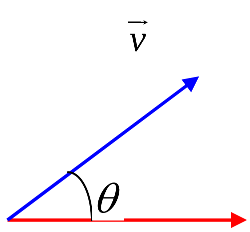
$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

$\vec{u}$ and $\vec{v}$ are orthogonal $\Leftrightarrow$ $\vec{u}\cdot\vec{v}=0$

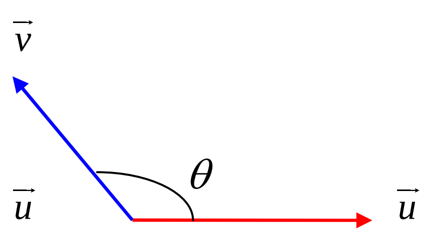
Observation

why?

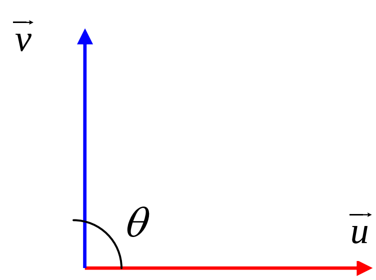
- $\vec{u}\cdot\vec{v}>0 \Rightarrow$ angle between $\vec{u}$ and $\vec{v}$ is acute
- $\vec{u}\cdot\vec{v}<0 \Rightarrow$ angle between $\vec{u}$ and $\vec{v}$ is obtuse
- $\vec{u}\cdot\vec{v}=0 \Rightarrow$ $\vec{u}$ and $\vec{v}$ are orthogonal



$\vec{u}\cdot\vec{v}>0$



$\vec{u}\cdot\vec{v}<0$



$\vec{u}\cdot\vec{v}=0$

Question 7(a)/813: Use vectors to show that A(2,-1,1), B(3,2,-1) and C(7,0,-2) are vertices of a right triangle. At which vertex is the right angle?

Question 10/814: Find two unit vectors in 2-space that make an angle of 45° with $4\vec{i}+3\vec{j}$.

Question 35/815: Find, to the nearest degree, acute angle formed by two diagonals of a cube.

Direction Angles and Direction Cosines of a Vector

Given a vector $\vec{v}=\langle v_1,v_2,v_3\rangle$

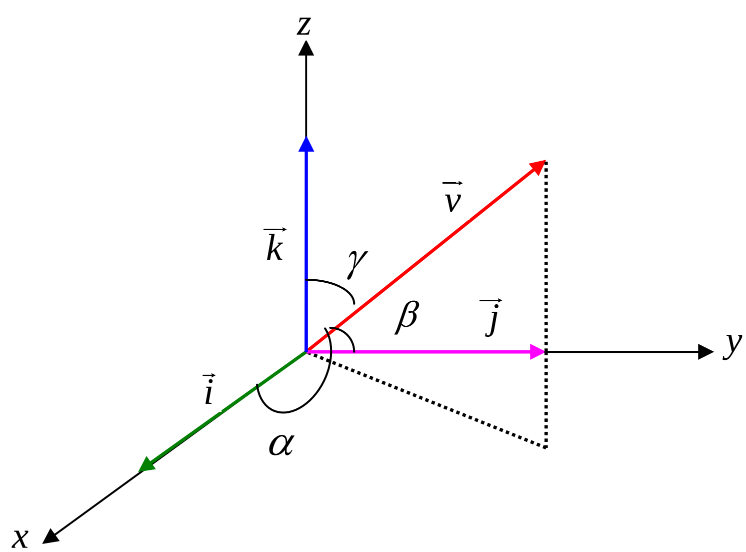
- Angles α,β,γ between $\vec{v}$ and vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ (i.e., between $\vec{v}$ and X-axis, Y-axis, Z-axis, respectively), are called ***direction angles***.
- The cosines $\cos\alpha,\cos\beta,\cos\gamma$ of direction angles are called ***direction cosines*** of $\vec{v}$.

The ***direction cosines*** of $\vec{v}=\langle v_1,v_2,v_3\rangle$ are

$$\cos\alpha=\frac{\vec{v}\cdot\mathbf{i}}{\|\vec{v}\|\|\mathbf{i}\|}=\frac{v_1}{\|\vec{v}\|}$$

$$\cos\beta=\frac{\vec{v}\cdot\mathbf{j}}{\|\vec{v}\|\|\mathbf{j}\|}=\frac{v_2}{\|\vec{v}\|}$$

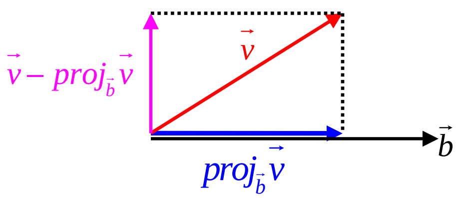
$$\cos\gamma=\frac{\vec{v}\cdot\mathbf{k}}{\|\vec{v}\|\|\mathbf{k}\|}=\frac{v_3}{\|\vec{v}\|}$$

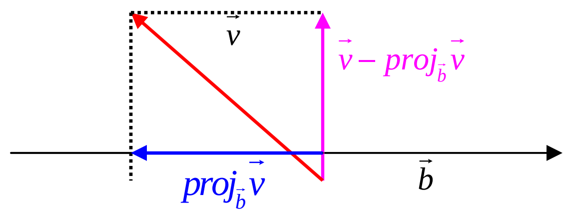


Projection of a Vector on another Vector

Q. What is projection of $\vec{v}$ on $\vec{b}$ (geometrically)?

A. Roughly speaking shadow of $\vec{v}$ on $\vec{b}$ (see figures below & explanation in class)

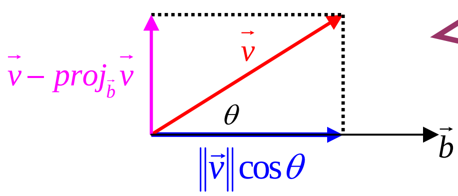




Using dot product to find projection of a vector on another

Let θ be angle between $\vec{v}$ and $\vec{b}$.

Then



- Since $\|\vec{v}\|\cos\theta = \dfrac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|}$
- Hence $proj_{\vec{b}}\vec{v} = \left(\dfrac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|}\right)\left(\dfrac{\vec{b}}{\|\vec{b}\|}\right)$

The projection of $\vec{v}$ on $\vec{b}$ is given by $$proj_{\vec{b}}\vec{v} = \left(\frac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$$

$proj_{\vec{b}}\vec{v} = \left(\dfrac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|}\right)\left(\dfrac{\vec{b}}{\|\vec{b}\|}\right)$ = vector component of $\vec{v}$ along $\vec{b}$.

$\vec{v} - proj_{\vec{b}}\vec{v}$ = vector component of $\vec{v}$ orthogonal to $\vec{b}$

<u>Question 16(a)/814</u>: Find the direction cosines of $\vec{v} = 3\vec{i} - 2\vec{j} - 6\vec{k}$.

<u>Question 21(c)/814</u>: Find the vector component of $\vec{v} = -3\vec{i} - 2\vec{j}$ along $\vec{b} = 2\vec{i} + \vec{j}$ and the vector component of $\vec{v}$ orthogonal to $\vec{b}$.

Solve solved examples 1—6 and Questions 1—4, 7—25, 35—42.