12.2 Vectors

Basic Definitions

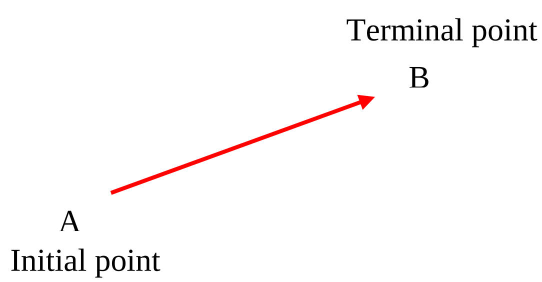
- **Scalar quantity:** A physical quantity which has only a magnitude is called scalar
- **Vector quantity:** A physical quantity which has a magnitude and a direction is called vector

Scalar quantities —— area, length, mass and temperature

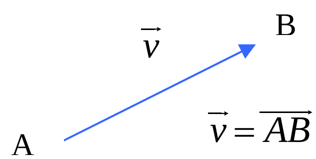
Vector quantities —— force and displacement

- **Geometric representation of vectors**

By arrows (as shown in figure)



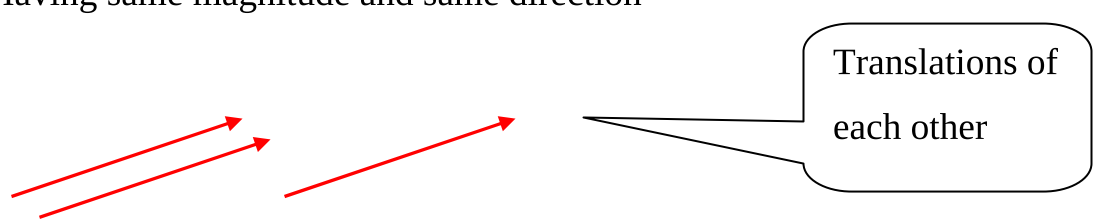
- length of arrow = magnitude of vector
- direction of arrow = direction of vector



If the initial point of $\vec{v}$ is A and the terminal point is B, then we write $\vec{v} = \overrightarrow{AB}$ when we want to emphasize the initial and terminal points.

- **Equivalent vectors**

Having same magnitude and same direction



- **Zero vector $\vec{0}$**

  Vector of length zero.

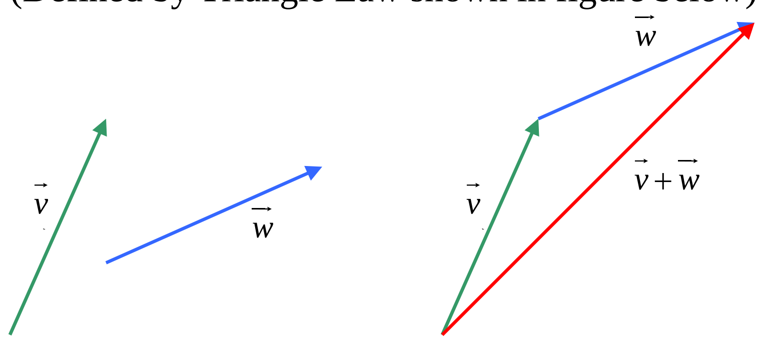
  Has no specific direction

Some Operations on Vectors

- **Vector addition**

  If $\vec{v}$ and $\vec{w}$ are vectors, then the sum $\vec{v}+\vec{w}$ is the vector from the initial point of $\vec{v}$ to the terminal point of $\vec{w}$ when the vectors are positioned so the initial point of $\vec{w}$ is at the terminal point of $\vec{v}$.

  (Defined by Triangle Law shown in figure below)



See explanation in class

- **Scalar multiplication**

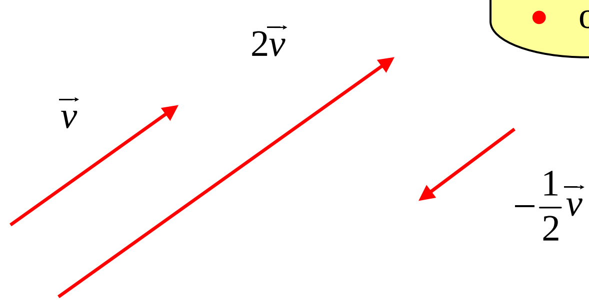
  Given a vector $\vec{v}$ and a scalar k . Then

  $k\vec{v} =$ a vector with

  length $|k|$ times the length of $\vec{v}$

  direction
  - same as $\vec{v}$ if $k > 0$
  - opposite to $\vec{v}$ if $k < 0$

Example:



- **Negative of a vector**

  $-\vec{v} = (-1)\vec{v}$ is called negative of $\vec{v}$

  Have same magnitude
  but opposite direction

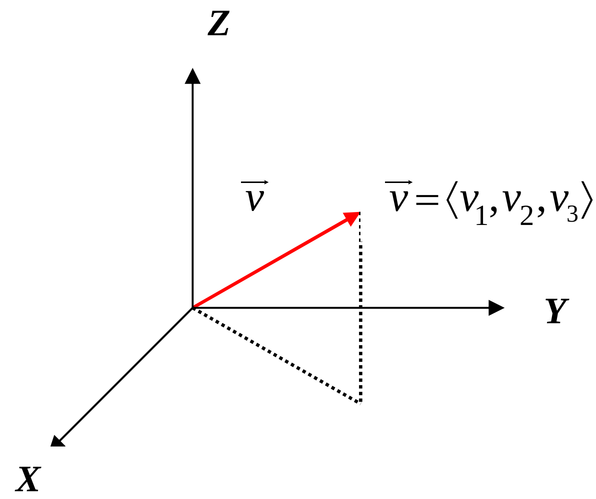
- **Difference of two vectors**

  $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$

Component Form of Vectors

- **Components of a vector with initial point $O(0,0,0)$ and terminal point $P(v_1,v_2,v_3)$**

$$\vec{v}=\langle v_1,v_2,v_3\rangle$$



- **Components of a vector with initial point $P_1(x_1,y_1,z_1)$ and terminal point $P_2(x_2,y_2,z_2)$**

$$\vec{v}=\overrightarrow{P_1P_2}=\langle x_2-x_1,y_2-y_1,z_2-z_1\rangle$$

- **Components of zero vector**

$$\langle 0,0,0\rangle$$

- **Equivalent vectors**

Corresponding components are same

Arithmetic Operations on Vectors (in components)

If $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$ then

- $\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$
- $\vec{v} - \vec{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle$
- $k\vec{v} = \langle kv_1, kv_2, kv_3 \rangle$

Rules of Vector Arithmetic

For any vector $\vec{u}$, $\vec{v}$ and $\vec{w}$ and any scalars k and l , the following relations hold:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{0} + \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $k(l\vec{u}) = (kl)\vec{u}$
- $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$
- $(k + l)\vec{u} = k\vec{u} + l\vec{u}$

Given $\vec{v} = \langle v_1, v_2, v_3 \rangle$ then the ***magnitude*** or ***length*** or ***norm*** of $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Note: $\|c\vec{v}\| = |c|\|\vec{v}\|$.

Why?

A vector whose length is 1 is called a ***unit vector***.

Finding a unit vector in the direction of a given vector is called ***normalization***.

How to normalize a vector

Given $\vec{v}$.

Then $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$ is a unit vector in the direction of $\vec{v}$.

In an ***XYZ***-coordinate system the ***unit vector*** along the ***X***-, ***Y***- and ***Z***- axes are denoted by $\vec{i}$, $\vec{j}$ and $\vec{k}$, respectively.

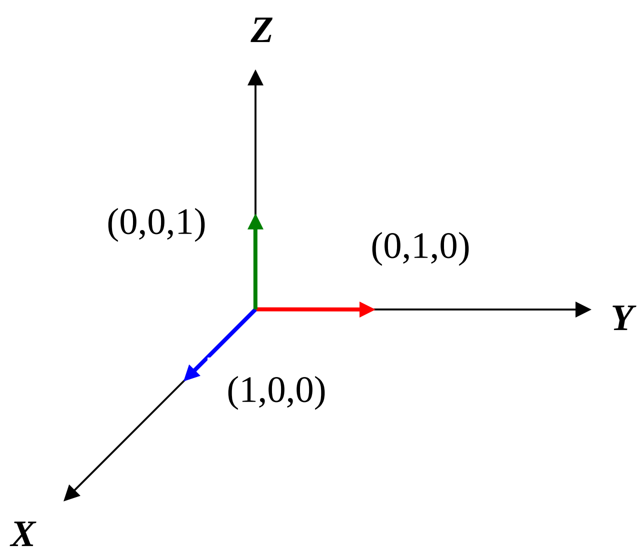
$\mathbf{i} = \langle 1, 0, 0 \rangle$: unit vector along X-axis

$\mathbf{j} = \langle 0, 1, 0 \rangle$: unit vector along Y-axis

$\mathbf{k} = \langle 0, 0, 1 \rangle$: unit vector along Z-axis

Any vector in 3-space can be expressed in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$

$$\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\langle 1, 0, 0 \rangle + v_2\langle 0, 1, 0 \rangle + v_3\langle 0, 0, 1 \rangle = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

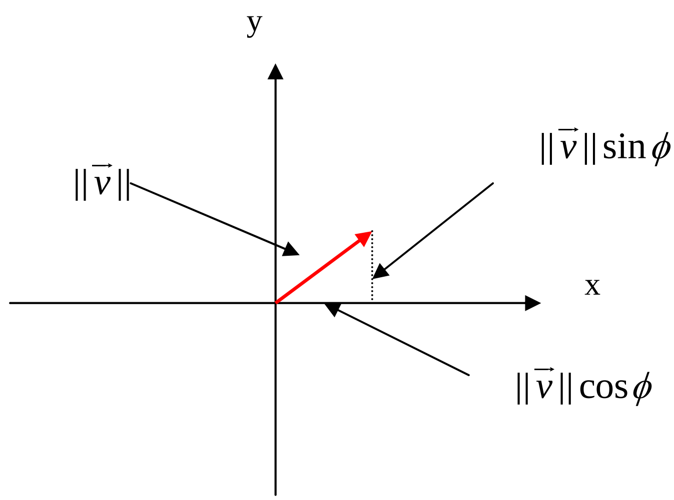


- From, we have done the concepts and methods for vectors in 3-space.
- All this holds for vectors in 2-space in a similar fashion.

Question 8(a)/804 Given $P_1(-6,-2)$, $P_2(-4,-1)$. Find the components of the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_1}$. Are the two vectors equivalent?

<u>Question 10(b)/804</u>: Find the terminal point of $\vec{v} = \vec{i} + 2\vec{j} - 3\vec{k}$ if the initial point is (-2, 1, 4).

Finding a vector in 2-space if its length and angle with X-axis are given:



- Given the length $\|\vec{v}\|$ a vector $\vec{v}$ which makes angle ϕ with positive X-axis.
- Then $\vec{v}$ is given in component form as

$$\vec{v} = \left\langle \|\vec{v}\|\cos\phi, \|\vec{v}\|\sin\phi \right\rangle$$

Example 12.2.5 Find a vector $\vec{v}$ that makes angle $\frac{\pi}{3}$ with the $\boldsymbol{X}$-axis and has magnitude $\|\vec{v}\| = 4$.

Solution:

$$\vec{v} = \left\langle \|\vec{v}\|\cos\frac{\pi}{3}, \|\vec{v}\|\sin\frac{\pi}{3} \right\rangle$$

$$\Rightarrow \quad \vec{v} = \left\langle 4\cdot\frac{1}{2}, 4\cdot\frac{\sqrt{3}}{2} \right\rangle = \left\langle 2, 2\sqrt{3} \right\rangle.$$

Finding a vector if its length and direction are known:

If $\vec{u}$ is a unit vector in direction of $\vec{v}$ then

$$\vec{v} = \|\vec{v}\|\vec{u}$$

$\vec{v}$ is equal to its length times a unit vector in the same direction

Example 12.2.6 Find a vector $\vec{v}$ of length $\sqrt{5}$ in the direction of $\vec{w} = 2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$.

Question 17/804: Find unit vectors that satisfy the stated conditions.

(a) Same direction as $-\vec{i} + 4\vec{j}$.

(b) Opposite direction to $6\vec{i} - 4\vec{j} + 2\vec{k}$.

(c) Same direction as the vector from the point $A(-1, 0, 2)$ to the point $B(3,1,1)$.

Question 35/805: In each part, find two unit vectors in 2-space that satisfy the stated condition.

(a) Parallel to the line $y = 3x + 2$.

(b) Parallel to the line $x + y = 4$.

(c) Perpendicular to the line $y = -5x + 1$.