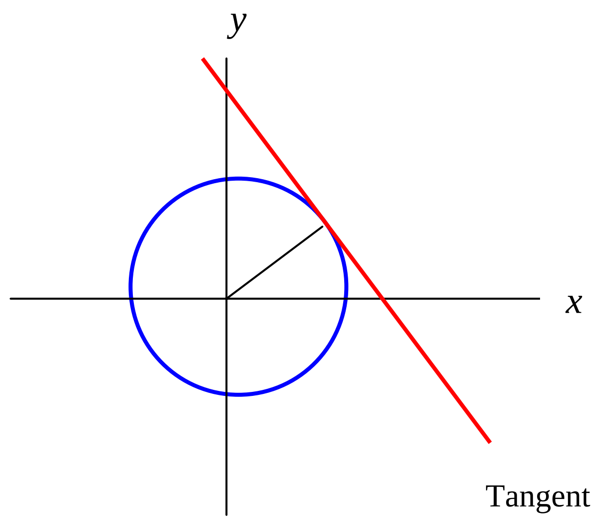
11.2 Tangent Lines and Arc Lengths for Parametric & Polar Curves

Tangent Lines to Parametric Curves



Consider the parametric equations of a curve

$$x = f(t),\ \ y = g(t); \qquad a \le t \le b$$

- $\dfrac{dy}{dx}$ is defined as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \text{ provided } \frac{dx}{dt} \neq 0$$

- Similarly

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

* slope of tangent line at $x = f(t_0),\ y = g(t_0)$

$$\left.\frac{dy}{dx}\right|_{t=t_0}$$

* Tangent is horizontal when

$$\frac{dy}{dt} = 0 \text{ and } \frac{dx}{dt} \neq 0$$

* Tangent is vertical when

$$\frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} \neq 0$$

* Points where $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$ are called singular points

Question 10/741 Consider the parametric equations $x = \cos\phi,\ y = 3\sin\phi$. Find

$$\frac{dy}{dx}, \frac{d^2y}{dx^2} \text{ at } \phi = 5\pi/6.$$

Question 11/741 **(a)** Find the equation of the tangent line to the curve

$x = e^t,\ y = e^{-t}$ at $t = 1$ without eliminating the parameter.

(b) Check your answer by eliminating the parameter.

Question 13/742 Find all values of t at which the parametric curve $x = 2\cos t,\ y = 4\sin t\ (0 \le t \le 2\pi)$ has (a) a horizontal tangent line and (b) a vertical tangent line.

Tangent Lines to Polar Curves

Every polar equation $r = f(\theta)$ can be regarded as a parametric equation (with parameter θ)

$$x = r\cos\theta \quad \left(\text{or } x = f(\theta)\cos\theta\right)$$
$$y = r\sin\theta \quad \left(\text{or } y = f(\theta)\sin\theta\right)$$

From which we obtain

$$\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$$
$$\frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta$$

- Then as above

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}$$

* Tangent is horizontal when $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$
* Tangent is vertical when $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$
* Points where $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} = 0$ are called singular points

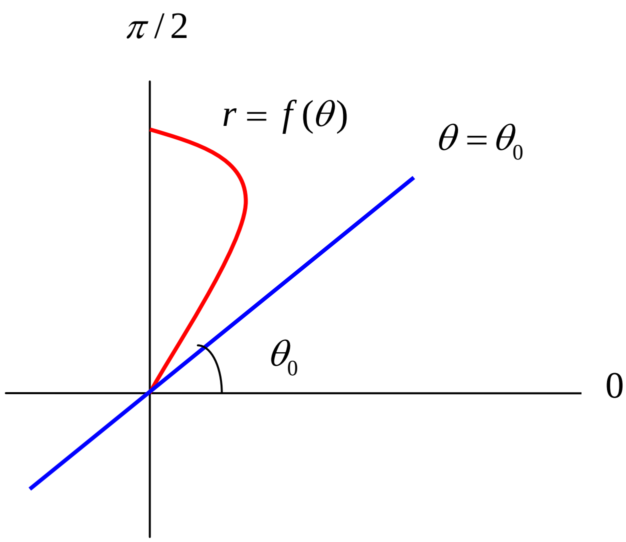
Question 25/742 Find the slope of the tangent line to the polar curve $r = \cos 3\theta$ at $\theta = 3\pi/4$.

Question 29/742 Find polar coordinates of all points at which the polar curve $r = a(1+\cos\theta)$ has a horizontal or a vertical tangent line.

Tangent Lines to Polar Curves at Pole

If θ_0 is a value of θ for which $r = f(\theta) = 0$ and $\frac{dr}{d\theta} \neq 0$ at $\theta = \theta_0$ then $\theta = \theta_0$ is tangent to $r = f(\theta)$ at origin.

To find all the tangents to $r = f(\theta)$ at origin, we need to **find all values of θ for which $f(\theta) = 0$.**



<u>Question 35/742</u> Sketch the polar curve and find polar equation of the tangent lines to the curve $r = 4\sqrt{\cos 2\theta}$ at the pole.

Arc Length Formula for Polar Curve

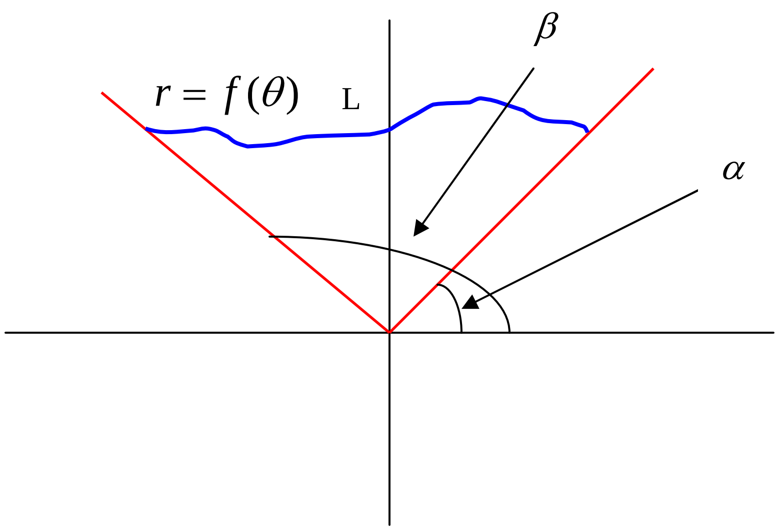
Given a polar curve $r = f(\theta)$

(θ increases from α to θ)

The arc length of the curve from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

We assume that curve is traced only once and $\alpha \le \theta \le \beta$



Question 44/742 Calculate the arc length of the curve $r = \sin^3(\theta/3)$ from $\theta = 0$ to $\theta = \pi/2$.

Surface Generated by Revolving the Curve

Given parametric equations of a curve

$$x = f(t), \quad y = g(t) \quad (a \le t \le b)$$

then the area of the surface generated by revolving this curve about X-axis is

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

and the area of the surface generated by revolving this curve about Y-axis is

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

We assume that curve is traced only once and $f'(t)$ and $g'(t)$ are continuous

Question 50/743 Find the area of the surface generated by revolving the curve

$$x = e^t \cos t, \quad y = e^t \sin t \quad (0 \le t \le \pi/2)$$

about X-axis.

Solve all the solved questions given in the book and Questions

1—14, 21—30, 33—44, 49—54.