Solution of Important Homework Problem

Q.34-(20.1): By Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$,
therefore $(\cos\theta + i\sin\theta)^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

Q.1-(20.2): $z = x + iy$
$|z - 8 + 4i| = |z - (8 - 4i)| = 9$ is a circle of radius 9 and center at $8 - 4i$.

Q.2-(20.2): $|z| = |x + iy| = \sqrt{(x^2 + y^2)}$ and $|z - i| = |x + iy - i| = \sqrt{(x^2 + y^2 - 2y + 1)}$
$x^2 + y^2 = x^2 + y^2 - 2y + 1$
$y = \frac{1}{2}$, which is a vertical line.

Q.16-(20.2): $1 < |z| \le 4$ or $1 < x^2 + y^2 \le 16$, is the annular region lying between two concentric circles of radius 1 and 4.

Q.37-(20.2): Find the limit of $\left\{\frac{1 + 3n^2 i}{2n^2 - n}\right\}$ as $n \to \infty$.

$$\lim_{n\to\infty}\frac{1 + 3n^2 i}{2n^2 - n} = \lim_{n\to\infty}\frac{1}{2n^2 - n} + i\lim_{n\to\infty}\frac{3n^2}{2n^2 - n} = 0 + \frac{3}{2}i$$

Q.2-(21.1): $f(z) = z^2 - iz = x^2 + 2ixy - y^2 - ix + y = (x^2 - y^2 + y) + i(2xy - x)$
$u(x,y) = x^2 - y^2 + y,\ v(x,y) = 2xy - x$

$$\frac{\partial u(x,y)}{\partial x} = 2x,\ \frac{\partial u(x,y)}{\partial y} = -2y + 1,\ \frac{\partial v(x,y)}{\partial x} = 2y - 1,\ \frac{\partial v(x,y)}{\partial y} = 2x$$

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} = 2x, \text{ and } \frac{\partial v(x,y)}{\partial x} = -\frac{\partial u(x,y)}{\partial y} = 2y - 1$$

CR equations are satisfied every where, therefore f is differentiable every where.

Q.12-(21.1): $f(z) = \frac{z - i}{z + i} = \frac{x + iy - i}{x + iy + i} = \frac{x + i(y - 1)}{x + i(y + 1)} = \frac{x^2 + y^2 - 1 - 2xi}{x^2 + (y + 1)^2}$

$$u(x,y) = \frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2}, \text{and } v(x,y) = \frac{-2x}{x^2 + (y + 1)^2}$$

$$\frac{\partial u(x,y)}{\partial x} = 4x\frac{y + 1}{(x^2 + y^2 + 2y + 1)^2},\quad \frac{\partial u(x,y)}{\partial y} = 2\frac{y^2 + 2y - x^2 + 1}{(x^2 + y^2 + 2y + 1)^2},$$

$$\frac{\partial v(x,y)}{\partial x} = -2\frac{y^2 + 2y - x^2 + 1}{(x^2 + y^2 + 2y + 1)^2},\quad \frac{\partial v(x,y)}{\partial y} = 4x\frac{y + 1}{(x^2 + y^2 + 2y + 1)^2}$$

CR equations are satisfied and $u(x,y)$ and $v(x,y)$ are continuous every where except at

$x = 0, y = -1$ or $z = -i$. Thus $f(z)$ is differentiable at all points except at $z = -i$.

Q.3-(21.2): Determine radius of convergence of $\sum_{n=0}^{\infty}\left(\frac{n^n}{(n + 1)^n}\right)(z - 1 + 3i)^n$

$$R = \lim_{n\to\infty}\left|\frac{(n + 1)^{n+1}}{(n + 2)^{n+1}}(z - 1 + 3i)^{n+1}\frac{(n + 1)^n}{n^n}\frac{1}{(z - 1 + 3i)^n}\right| = \lim_{n\to\infty}\left|\frac{(n + 1)^{n+1}}{(n + 2)^{n+1}}\frac{(n + 1)^n}{n^n}(z - 1 + 3i\right.$$

$$= \lim_{n\to\infty}\left(\frac{n + 1}{n + 2}\right)^{n+1}\left(\frac{n + 1}{n}\right)^n|(z - 1 + 3i)| = \lim_{n\to\infty}\left(\frac{n + 1}{n + 1 + 1}\right)^{n+1}\left(\frac{n + 1}{n}\right)^n|(z - 1 + 3i)|$$

$$= \lim_{n\to\infty}\frac{1}{\left(\frac{n + 1 + 1}{n + 1}\right)^{n+1}}\lim_{n\to\infty}\left(\frac{n + 1}{n}\right)^n|(z - 1 + 3i)| = \lim_{n\to\infty}\frac{1}{\left(1 + \frac{1}{n + 1}\right)^{n+1}}\left(1 + \frac{1}{n}\right)^n|(z - 1 +$$