Solution Math 301-103 Sec: 03 Quiz 3 (B)

Q.1: Find the surface area of portion of the paraboloid $3z = 2x^2 + 2y^2$ that is below the plane $z = 4$.

Sol: We need to evaluate $\iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\ dA$, where $z = f(x, y) = \frac{2}{3}x^2 + \frac{2}{3}y^2$ and $f_x(x, y) = \frac{4}{3}x,\ f_y(x, y) = \frac{4}{3}y$, R is the region of the projection of the paraboloid onto xy-plane.

At $z = 4$, $3z = 2x^2 + 2y^2 \Rightarrow x^2 + y^2 = 6$, a circle of radius $\sqrt{6}$. Therefore the projection of the paraboloid onto xy-plane is the circular disk $x^2 + y^2 = 6$.

$$A(S) = \iint_R \sqrt{1 + \tfrac{16}{9}x^2 + \tfrac{16}{9}y^2}dA = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{1 + \tfrac{16}{9}r^2}\ r\ dr\ d\theta = \frac{3\pi}{8}\left((\tfrac{35}{3})^{3/2} - 1\right).$$

Q.2: Use the divergence theorem to evaluate $\iint_S (\vec{F}.\hat{n})\ dS$ where $Div(F) = \dfrac{1}{x^2 + y^2 + z^2}$ and D the region bounded by $x^2 + y^2 + z^2 = 25$, $x^2 + y^2 + z^2 = 16$.

Sol: $\iint_S (\vec{F}.\hat{n})\ dS = \iiint_D DivF dV = \iiint_D \frac{1}{x^2+y^2+z^2} dV$

$$= \int_0^{2\pi} \int_0^{\pi} \int_3^5 \tfrac{1}{\rho^2}\rho^2\ \sin(\phi)\ d\rho\ d\phi\ d\theta = (2\pi)(2)(5-3) = 8\pi.$$

Q.3: Use Stokes's theorem to evaluate $\oint_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = 3z\hat{\imath} - 2x\hat{\jmath} + 4y\hat{k}$ and S that portion of paraboloid $z = 25 - x^2 - y^2$ for $z \geq 0$.

Sol: Curl $\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & -3x & 2y \end{vmatrix} = 2\hat{\imath} + 2\hat{\jmath} - 3\hat{k}.$

$g(x, y, z) = x^2 + y^2 + z - 16 = 0$, $\nabla g = \langle 2x, 2y, 1 \rangle$ and $\hat{n} = \frac{\langle 2x, 2y, 1\rangle}{\sqrt{4x^2+4y^2+1}}$

$z = f(x, y) = 16 - x^2 - y^2 \Rightarrow f_x = -2x,\ f_y = -2y$ and $dS = \sqrt{1 + 4x^2 + 4y^2}\ dA$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S Curl\vec{F} \cdot \hat{n}\ dS = \iint_R \frac{4x+4y-3}{\sqrt{1+4x^2+4y^2}} \sqrt{1 + 4x^2 + 4y^2}\ dA$$

$$= \int_0^{2\pi} \int_0^4 [4r\cos(\theta) + 4r\sin(\theta) - 3]\ r\ dr\ d\theta = -48\pi$$