King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 302 Major Exam 2
The Summer Semester of 2009-2010 (093)

Q:1 (10 points) Evaluate the surface integral $\iint_{\Sigma} f(x,y,z)d\sigma$, where $f(x,y,z) = 2x+3y$ and $\sum$ is the part of the plane $2x+4y+5z=10$ lying above the triangle in the $xy-$plane having vertices $(0,0)$, $(2,0)$ and $(2,1)$.

Sol : $2x+4y+5z=10 \Rightarrow z = -\frac{2}{5}x - \frac{4}{5}y + 2$ and $\frac{\partial z}{\partial x} = -\frac{2}{5}$, $\frac{\partial z}{\partial y} = -\frac{4}{5}$

$$d\sigma = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA = \sqrt{1+\frac{4}{25}+\frac{16}{25}} = \frac{3}{5}\sqrt{5}dA$$

$$\iint_{\Sigma} f(x,y,z)d\sigma = \iint_{\Sigma} (2x+3y)\, d\sigma = \iint_{R} (2x+3y)\, \frac{3}{5}\sqrt{5}dA$$

$$= \frac{3}{5}\sqrt{5}\int_0^2 \int_0^{\frac{x}{2}} (2x+3y)\, dydx = \frac{11}{5}\sqrt{5}.$$

Q:2 (10 points) Use divergence theorem to evaluate $\iint_{\Sigma} \vec{F}.N\ d\sigma$, where $\vec{F}(x,y,z) = 3x^2\mathbf{i} + 2y^2\mathbf{j} + \frac{3}{2}z^2\ \mathbf{k}$ and $\sum$ the cone $z = \sqrt{x^2+y^2}$ for $x^2+y^2 \le 4$, together with the cap consisting of points $(x,y,2)$ with $x^2+y^2 \le 4$.

Sol : Divergence Theorem $\iint_{\Sigma} \vec{F}.N\ d\sigma = \iiint_{M} div\,(F)\, ddV$

$$div\,(F) = \frac{\partial\,(3x^2)}{\partial x} + \frac{\partial\,(2y^2)}{\partial y} + \frac{\partial\left(\frac{3}{2}z^2\right)}{\partial z} = 6x+4y+3z$$

$$\iiint_{M} div\,(F)\, ddV = \iiint_{M} (6x+4y+3z)\, dV$$

$$= \int_0^{2\pi}\int_0^2\int_r^2 (6r\cos\theta + 4r\sin\theta + 3z)\, rdzdrd\theta = 12\pi.$$

Q:3 (12 points) Use Stokes's theorem to evaluate $\oint_C \vec{F} \cdot d\vec{R}$, where $\vec{F}(x, y, z) = 2z\ \mathbf{i} - x\ \mathbf{j} + 3y\ \mathbf{k}$ and $\sum$ the cone $z = \sqrt{x^2 + y^2}$ for $0 \le z \le 4$.

Sol : Stokes Theorem $\oint_C \vec{F} \cdot d\vec{R} = \iint_{\Sigma} \left(\nabla \times \vec{F} \right) .N\ d\sigma$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & -x & 3y \end{vmatrix} = i(3) - j(-2) + k(-1) = 3i + 2j - k$$

To find N, let $g(x, y, z) = \sqrt{x^2 + y^2} - z = 0$.

Then $\nabla g = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle$ and $\|\nabla g\| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} = \sqrt{2}$

So $N = \left\langle \frac{x}{\sqrt{2}\sqrt{x^2+y^2}}, \frac{y}{\sqrt{2}\sqrt{x^2+y^2}}, \frac{-1}{\sqrt{2}} \right\rangle$.

Also $d\sigma = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} dA = \sqrt{2} dA$

$$\left(\nabla \times \vec{F} \right) .N = \frac{3x}{\sqrt{2}\sqrt{x^2+y^2}} + \frac{2y}{\sqrt{2}\sqrt{x^2+y^2}} + \frac{1}{\sqrt{2}}$$

$$\iint_{\Sigma} \left(\nabla \times \vec{F} \right) .N\ d\sigma = \iint_{\Sigma} \left(\frac{3x}{\sqrt{2}\sqrt{x^2+y^2}} + \frac{2y}{\sqrt{2}\sqrt{x^2+y^2}} + \frac{1}{\sqrt{2}} \right) d\sigma = \iint_R \left(\frac{3x+2y}{\sqrt{x^2+y^2}} + 1 \right) dA$$

$$= \int_0^{2\pi} \int_0^4 (3\cos\theta + 2\sin\theta + 1)\, r dr d\theta = 16\pi.$$

Q:4 (a) (5 points) Let z and w be complex numbers such that $\bar{z}w \neq 1$ but such that either z or w has magnitude 1. Prove that $\left| \frac{z - w}{1 - \bar{z}w} \right| = 1$.

(b) (5 points) Transform into cartesian equation and sketch its graph $|z + 2 - 3i| = |z - 4 + 7i|$

Sol : (a) Let $|z| = 1$, then $|\bar{z}| = 1$ and $|z\bar{z}| = 1$

$$\left| \frac{z - w}{1 - \bar{z}w} \right| = \left| \frac{z - w}{z\bar{z} - \bar{z}w} \right| = \left| \frac{z - w}{\bar{z}(z - w)} \right| = \frac{1}{|\bar{z}|} \left| \frac{z - w}{(z - w)} \right| = 1$$

Now let $|w| = 1$, then $|\bar{w}| = 1$ and $|w\bar{w}| = 1$

$$\left| \frac{z - w}{1 - \bar{z}w} \right| = \left| \frac{z - w}{w\bar{w} - \bar{z}w} \right| = \left| \frac{z - w}{w(\bar{w} - \bar{z})} \right| = \frac{1}{|w|} \left| \frac{z - w}{(\bar{w} - \bar{z})} \right| = \left| \frac{z - w}{\bar{z} - \bar{w}} \right| = 1 \text{ because } |Z| = \left| \bar{Z} \right|.$$

$(b)\ |z+2-3i|^2 = |z-4+7i|^2$

$$\Rightarrow (x+2+i(y-3))(x+2-i(y-3)) = (x-4+i(y+7))(x-4-i(y+7))$$

$$x^2+4x+4+y^2-6y+9 = x^2-8x+16+y^2+14y+49$$

$$12x-52=20y \ \Rightarrow y=\frac{12}{20}x-\frac{52}{20}=\frac{3}{5}x-\frac{13}{5},$$

a line with slope $m=\frac{3}{5}$ and $y-intercept=-\frac{13}{5}$.

Q:5 .(10 points) Let $f(z)=-4z+\frac{1}{z}$. Write $f(z)=u(x,y)+iv(x,y)$ and determine all points at which the Cauchy-Riemann equations are satisfied and determine all points at which f is differentiable.

Sol : $f(z)=-4(x+iy)+\dfrac{1}{x+iy}=-4x-4iy+\dfrac{x-iy}{x^2+y^2}=-4x+\dfrac{x}{x^2+y^2}+i\left(-4y-\dfrac{y}{x^2+y^2}\right)$

$u(x,y)=-4x+\dfrac{x}{x^2+y^2}$ and $v(x,y)=-4y-\dfrac{y}{x^2+y^2}$

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(-4x+\frac{x}{x^2+y^2}\right)=\frac{1}{x^2+y^2}-2\frac{x^2}{x^4+y^4+2x^2y^2}-4=\frac{y^2-x^2}{(x^2+y^2)}-4$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(-4x+\frac{x}{x^2+y^2}\right)=-2x\frac{y}{x^4+y^4+2x^2y^2}$$

$$\frac{\partial v}{\partial x}=\frac{\partial}{\partial x}\left(-4y-\frac{y}{x^2+y^2}\right)=2x\frac{y}{x^4+y^4+2x^2y^2}$$

$$\frac{\partial v}{\partial y}=\frac{\partial}{\partial y}\left(-4y-\frac{y}{x^2+y^2}\right)=2\frac{y^2}{x^4+y^4+2x^2y^2}-\frac{1}{x^2+y^2}-4=\frac{y^2-x^2}{(x^2+y^2)}-4$$

Cauchy-Riemann equations $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y}=-\dfrac{\partial u}{\partial x}$ are true for all points except $z=0\ (x=0,y=0)$

Therefore f is differentiable at all points except $z=0$.

Q:6 (a) (8 points) Determine radius of convergence and open disk of convergence of the power series $\sum_{n=0}^{\infty}\dfrac{n^n}{(n+1)^n}(z-2+3i)^n$.

(b) (5 points) Verify the identity $\cos(z+w)=\cos z\cos w-\sin z\sin w$ for complex numbers z and w.

Sol : $(a)\ \lim\limits_{n\to}\left|\dfrac{(n+1)^{n+1}(z-2+3i)^{n+1}(n+1)^{n+1}}{(n+2)^{n+1}\ \ n^n(z-2+3i)^n}\right|=\lim\limits_{n\to}\left(\dfrac{n+1}{n+2}\right)^{n+1}\left(\dfrac{n+1}{n}\right)^n|z-2+3i|$

$=\lim\limits_{n\to}\left(1-\dfrac{1}{n+2}\right)^{n+1}\left(1+\dfrac{1}{n}\right)^n|z-2+3i| = e^{-1}e\,|z-2+3i| = |z-2+3i| < 1$ for convergence

Radius of convergence $R = 1$ and open disk of convergence is $|z - 2 + 3i| < 1$.

$$(b)\ \cos z \cos w - \sin z \sin w = \frac{e^{iz} + e^{-iz}}{2}\frac{e^{iw} + e^{-iw}}{2} - \frac{e^{iz} - e^{-iz}}{2i}\frac{e^{iw} - e^{-iw}}{2i}$$

$$= \frac{e^{i(z+w)} + e^{i(z-w)} + e^{i(w-z)} + e^{-i(z+w)}}{4} - \frac{e^{i(z+w)} - e^{i(z-w)} - e^{i(w-z)} + e^{-i(z+w)}}{-4}$$

$$= \frac{2e^{i(z+w)} + 2e^{-i(z+w)}}{4} = \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} = \cos(z + w).$$

Q:7 (4 points) Find z such that $e^z = -1 + 2i$.

(b) (6 points) Find all values of $(1 + i)^{\frac{3}{5}}$.

Sol : (a) $e^z = -1 + 2i \ \Rightarrow z = \log(-1 + 2i) = \ln\left(\sqrt{5}\right) + i\left(\pi - \tan^{-1} 2 + 2n\pi\right)$

(b) Let $z = (1 + i)^3 = -2 + 2i$,then $|z| = \sqrt{8} = 2^{\frac{3}{2}}$ and $\theta = \dfrac{3\pi}{4} + 2k\pi$

So $z = 2^{\frac{3}{2}} e^{i\left(\frac{3\pi}{4} + 2k\pi\right)}$ and $(1 + i)^{\frac{3}{5}} = z^{\frac{1}{5}} = 2^{\frac{3}{10}} e^{i\left(\frac{3\pi}{20} + \frac{2k\pi}{5}\right)}$, $k = 0, 1, 2, 3, 4$

$k = 0,\ z_1 = 2^{\frac{3}{10}} e^{i\left(\frac{3\pi}{20}\right)} = 2^{\frac{3}{10}}\left(\cos\frac{3\pi}{20} + i \sin\cos\frac{3\pi}{20}\right)$

$k = 1,\ z_2 = 2^{\frac{3}{10}} e^{i\left(\frac{11\pi}{20}\right)} = 2^{\frac{3}{10}}\left(\cos\frac{11\pi}{20} + i \sin\cos\frac{11\pi}{20}\right)$

$k = 2,\ z_3 = 2^{\frac{3}{10}} e^{i\left(\frac{19\pi}{20}\right)} = 2^{\frac{3}{10}}\left(\cos\frac{19\pi}{20} + i \sin\cos\frac{19\pi}{20}\right)$

$k = 3,\ z_4 = 2^{\frac{3}{10}} e^{i\left(\frac{27\pi}{20}\right)} = 2^{\frac{3}{10}}\left(\cos\frac{27\pi}{20} + i \sin\cos\frac{27\pi}{20}\right)$

$k = 4,\ z_5 = 2^{\frac{3}{10}} e^{i\left(\frac{35\pi}{20}\right)} = 2^{\frac{3}{10}}\left(\cos\frac{35\pi}{20} + i \sin\cos\frac{35\pi}{20}\right)$