King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Solution Math 302 Major Exam I
The Summer Semester of 2009-2010 (093)

Q:1 Let $\mathbf{S}_a = \{(x, y, ax, a^2 - 5a + 6) \mid x, y, a \in \mathbf{R}\}$ be a subset of $\mathbf{R}^4$.

(a) (3 points) Find all values of a for which $\mathbf{S}_a$ is a subspace of $\mathbf{R}^4$.

Sol. For $O \in S_a$, $a^2 - 5a + 6 = 0 \Rightarrow (a-2)(a-3) = 0 \Rightarrow a = 2, 3$.

(b) (7 points) For each vale of a obtained in part (a), find a basis for $\mathbf{S}_a$.

Sol. For $a = 2$, $\mathbf{S}_2 = \{(x, y, 2x, 0) \mid x, y \in \mathbf{R}\}$ and $(x, y, 2x, 0) = x(1,0,2,0) + y(0,1,0,0)$

$\Rightarrow \{(1,0,2,0), (0,1,0,0)\}$ span S_2.

$\alpha(1,0,2,0) + \beta(0,1,0,0) = (0,0,0,0) \Rightarrow \alpha = 0$ and $\beta = 0$

$\Rightarrow \{(1,0,2,0), (0,1,0,0)\}$ is a basis for S_2

For $a = 3$, $\mathbf{S}_3 = \{(x, y, 3x, 0) \mid x, y \in \mathbf{R}\}$ and $(x, y, 3x, 0) = x(1,0,3,0) + y(0,1,0,0)$

$\Rightarrow \{(1,0,3,0), (0,1,0,0)\}$ span S_3.

$\alpha(1,0,3,0) + \beta(0,1,0,0) = (0,0,0,0) \Rightarrow \alpha = 0$ and $\beta = 0$

$\Rightarrow \{(1,0,3,0), (0,1,0,0)\}$ is a basis for S_3.

Q:2 (10 points) Find the general solution of the linear system

$$\begin{aligned} 3x - 2y + z &= 6 \\ x + 10y - z &= 2 \\ -3x - 2y + z &= 0 \end{aligned}$$

Sol. The system can be written in matrix form as $AX = b$,

where $A = \begin{bmatrix} 1 & 10 & -1 \\ 3 & -2 & 1 \\ -3 & -2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}$.

The augmented matrix $\left[\begin{array}{ccc|c} 1 & 10 & -1 & 2 \\ 3 & -2 & 1 & 6 \\ -3 & -2 & 1 & 0 \end{array}\right] \underset{3R_1+R_3}{\overset{-3R_1+R_2}{\longrightarrow}} \left[\begin{array}{ccc|c} 1 & 10 & -1 & 2 \\ 0 & -32 & 4 & 0 \\ 0 & 28 & -2 & 6 \end{array}\right]$

$$\xrightarrow{-\frac{1}{32}R_2} \left[\begin{array}{ccc|c} 1 & 10 & -1 & 2 \\ 0 & 1 & -\dfrac{1}{8} & 0 \\ 0 & 28 & -2 & 6 \end{array}\right] \xrightarrow{-28R_2+R_3} \left[\begin{array}{ccc|c} 1 & 10 & -1 & 2 \\ 0 & 1 & -\frac{1}{8} & 0 \\ 0 & 0 & \frac{3}{2} & 6 \end{array}\right]$$

$$\xrightarrow{\frac{2}{3}R_3} \left[\begin{array}{ccc|c} 1 & 10 & -1 & 2 \\ 0 & 1 & -\frac{1}{8} & 0 \\ 0 & 0 & 1 & 4 \end{array}\right] \underset{R_3+R_1}{\xrightarrow{\frac{1}{8}R_3+R_2}} \left[\begin{array}{ccc|c} 1 & 10 & 0 & 6 \\ 0 & 1 & 0 & \dfrac{1}{2} \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\xrightarrow{-10R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 4 \end{array}\right] \Rightarrow X = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 4 \end{bmatrix} \text{ is the solution.}$$

Q:3 Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

(a) (7 points) Find a matrix P that diagonalizes A. Also write $P^{-1}AP$.

Sol. To find eigenvalues, $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & -\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 0, 2, 3.$

For $\lambda = 0$, $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} \underset{-3R_3+R_2}{\xrightarrow{-2R_1+R_2}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow v_1 = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda = 2$, $\begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_3+R_2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow v_2 = \beta \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda = 3$, $\begin{bmatrix} -1 & 0 & 0 \\ 1 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow v_3 = \gamma \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$.

$P = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ and $P_{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Q:4 (10 points) Find equation of the tangent plane and normal line to the surface $2x - \cos(xyz) = 2$ at $\left(1, \dfrac{\pi}{2}, 1\right)$.

Sol. Let $g(x, y, z) = 2x - \cos(xyz) - 2 = 0$.

$\nabla g = \langle 2 + yz\sin(xyz), xz\sin(xyz), xy\sin(xyz) \rangle$ and $\nabla g\left(1, \frac{\pi}{2}, 1\right) = \langle 2 + \frac{\pi}{2}, 1, \frac{\pi}{2} \rangle$

Equation of tangent plane is $\left(2 + \frac{\pi}{2}\right)(x - 1) + 1\left(y - \frac{\pi}{2}\right) + \frac{\pi}{2}(z - 1) = 0.$

Equation of normal line is $\dfrac{x-1}{2+\frac{\pi}{2}}=\dfrac{y-\frac{\pi}{2}}{1}=\dfrac{z-1}{\frac{\pi}{2}}=t$

Q:5 Let $\mathbf{F}(x,y,z)=e^{xyz}\mathbf{i}+\ln(xyz)\ \mathbf{j}+\tan^{-1}(xyz)\ \mathbf{k}$.

(a) (5 points)Find $\nabla\cdot\mathbf{F}$

Sol. $\nabla\cdot F=\dfrac{\partial(e^{xyz})}{\partial x}+\dfrac{\partial(\ln(xyz))}{\partial y}+\dfrac{\partial(\tan^{-1}(xyz))}{\partial z}=yxe^{xyz}+\dfrac{1}{y}+\dfrac{xy}{1+(xyz)^2}$.

(b) (6 points)Find $\nabla\times\mathbf{F}$.

Sol. $$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ e^{xyz} & \ln(xyz) & \tan^{-1}(xyz)\end{vmatrix}$$

$$=\mathbf{i}\left(\frac{\partial(\tan^{-1}(xyz))}{\partial y}-\frac{\partial(\ln(xyz))}{\partial z}\right)+\mathbf{j}\left(\frac{\partial(e^{xyz})}{\partial z}-\frac{\partial(\tan^{-1}(xyz))}{\partial x}\right)+\mathbf{k}\left(\frac{\partial(\ln(xyz))}{\partial x}-\frac{\partial(e^{xyz})}{\partial y}\right)$$

$$=\mathbf{i}\left(\frac{xz}{1+(xyz)^2}-\frac{1}{z}\right)+\mathbf{j}\left(xye^{xyz}-\frac{yz}{1+(xyz)^2}\right)+\mathbf{k}\left(\frac{1}{x}-xze^{xyz}\right).$$

Q:6 (10 points) Evaluate the integral $\displaystyle\int_C (2x+3y)\,ds$, where the curve C is given by $x=y=\sqrt{z}$ for $0\le y\le 2$.

Sol. Let $y=t$, then $x=t$ and $z=t^2$ where $0\le t\le 2$, $\dfrac{dx}{dt}=1$, $\dfrac{dy}{dt}=1$ and $\dfrac{dz}{dt}=2t$

$$\int_C (2x+3y)\,ds=\int_0^2 5t\sqrt{1+1+4t^2}dt=\frac{5}{8}\int_0^2 8t\sqrt{2+4t^2}dt=\frac{5}{8}\frac{2}{3}\left(2+4t^2\right)^{\frac{3}{2}}\Big|_0^2$$

$$=\frac{5}{12}\left(18^{\frac{3}{2}}-2^{\frac{3}{2}}\right)=\frac{5.2\sqrt{2}}{12}(27-1)=\frac{65\sqrt{2}}{3}.$$

Q:7 (12 points) Let $\mathbf{F}(x,y)=2xy\ \mathbf{i}+3x^2\ \mathbf{j}$ and C is the boundary of the region determined by the graphs of $x=0,\ x^2+y^2=1,\ x\ge 0$. Use Greens' theorem to evaluate $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{R}$.

Sol The region is the right side half circle of radius 1 with center at $(0,0)$.

$$\oint_C \mathbf{F}\cdot d\mathbf{R}=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=\iint_R(6x-2x)\,dA=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\ r\cos\theta\ r\ dr\ d\theta=\frac{8}{3}.$$